The countable chain condition in products

Contents

1 Motivation

We have learned that the countable chain condition behaves unusually with respect to products. In fact, we learned that the question of whether the ccc is finitely productive is independent of the usual ZFC axioms. This means that given an arbitrary collection of ccc spaces, there is no hope of deciding whether their product is ccc.

With that said, there are still some partial results that are available to us. The first section of this note will outline the most important one of these, and some of its corollaries. The results in this section do not rely on any additional set-theoretic assumptions. They do rely on understanding the definition of a general (i.e., infinite) product of topological spaces, however. These are introduced in section 14 of the lecture notes. Students are welcome to do some independent research to learn these definitions if they want to understand this note before we get to that section.

The second section of this note will outline, in relatively simple terms, why the following statement is consistent with the usual ZFC axioms.

(1) Any finite product of ccc spaces is ccc.

In fact, a significant strengthening of this statement is consistent with ZFC—*any (not necessarily finite)* product of ccc spaces is ccc. This is the final result in the second section. In addition to needing to know how arbitrary products of topological spaces are defined, the reader will need to know what partial orders and ω_1 are in order to fully understand this section. That material is covered in section 10 of the lecture notes.

At the time of this writing, this note does not outline the proof that the Continuum Hypothesis implies the existence of a ccc space whose square is not ccc. This result is the usual way one shows that the *negation* of (1) is consistent with ZFC. This proof is somewhat beyond the scope of the course, but I will add it if I can formulate a straightforward treatment.

2 A partial result about the productivity of the ccc

2.1 Preliminaries

Recall the following definitions and facts from throughout the course.

Definition 2.1. *A topological space $(X, \mathcal{T})$ is said to have the <u>countable chain condition</u> if any collection of pairwise disjoint, nonempty, open subsets of X is countable.*

Notation. Given a set A, we denote by $[A]^{<\omega}$ the set of all finite subsets of A. More formally:

$$[A]^{<\omega} := \{\, F \subseteq A \,:\, F \text{ is finite} \,\}.$$

(Note that we previously used the notation $\mathrm{Fin}(A)$ for this, but the notation here is more standard throughout set theory.)

Definition 2.2. *Let A be a set. A collection $\mathcal{A} \subseteq [A]^{<\omega}$ is called a* Δ-system *if there is a finite set $r \in [A]^{<\omega}$ (which may be empty) such that $a \cap b = r$ for all distinct $a, b \in \mathcal{A}$. The set r is called the* root *of the Δ-system.*

Theorem 2.3 (Δ-System Lemma)**.** *Let A be an uncountable set, and let $\mathcal{A} \subseteq [A]^{<\omega}$ be an uncountable collection of finite subsets of A. Then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ that is a Δ-system.*

2.2 Results

Theorem 2.4. *Let I be a nonempty indexing set, and let $\{ (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \}$ be a family of topological spaces such that $\prod_{\alpha \in F} X_\alpha$ has the countable chain condition for all $F \in [I]^{<\omega}$. Then $\prod_{\alpha \in I} X_\alpha$ has the countable chain condition.*

Proof. Let $X = \prod_{\alpha \in I} X_\alpha$ be the product of the given spaces, with its product topology. The theorem is tautological if I is finite. We first treat the case where I is uncountable.

Assume for the sake of contradiction that $\mathcal{U}$ is an uncountable collection of pairwise disjoint, nonempty, open subsets of X. We may assume without loss of generality that each of the sets in $\mathcal{U}$ are basic open subsets of the product topology.

By definition of the product topology, for each $U = \prod_{\alpha \in I} U_\alpha \in \mathcal{U}$, there is a finite set $F_U \subseteq I$ such that $U_\alpha \neq X_\alpha$ if and only if $\alpha \in F_U$.

First, we present a useful lemma:

Lemma 2.5. *Let $U = \prod_{\alpha \in I} U_\alpha$ and $V = \prod_{\alpha \in I} V_\alpha$ be disjoint, nonempty, basic open subsets of X. Then $F_U \cap F_V \neq \emptyset$, and moreover there is some $\alpha \in F_U \cap F_V$ such that $U_\alpha \cap V_\alpha = \emptyset$.*

Proof. Assume U and V are disjoint and nonempty. First, suppose for the sake of contradiction that $F_U \cap F_V = \emptyset$. Then we can pick...

- ...a point $a_\alpha \in U_\alpha$ for all $\alpha \in F_U$,
- ...a point $b_\alpha \in V_\alpha$ for all $\alpha \in F_V$, and
- ...a point $c_\alpha \in X_\alpha$ for all $\alpha \in I \setminus (F_U \cup F_V)$.

Then the point $x \in X$ defined by:

$$\pi_\alpha(x) = \begin{cases} a_\alpha & \alpha \in F_U \\ b_\alpha & \alpha \in F_V \\ c_\alpha & \text{otherwise} \end{cases}$$

is in $U \cap V$, contradicting the assumption that U and V are disjoint.

Next, let $F = F_U \cap F_V$ and suppose for the sake of contradiction that $U_\alpha \cap V_\alpha \neq \emptyset$ for all $\alpha \in F$. Then we can pick...

- ...a point $a_\alpha \in U_\alpha \cap V_\alpha$ for all $\alpha \in F$,
- ...a point $b_\alpha \in U_\alpha$ for all $\alpha \in F_U \setminus F$,
- ...a point $c_\alpha \in V_\alpha$ for all $\alpha \in F_V \setminus F$, and
- ...a point $d_\alpha \in X_\alpha$ for all $\alpha \in I \setminus (F_U \cup F_V)$.

Then the point $x \in X$ defined by

$$\pi_\alpha(x) = \begin{cases} a_\alpha & \alpha \in F \\ b_\alpha & \alpha \in F_U \setminus F \\ c_\alpha & \alpha \in F_V \setminus F \\ d_\alpha & \text{otherwise} \end{cases}$$

is in $U \cap V$, again contradicting the assumption that $U \cap V = \emptyset$. This completes the proof of the lemma. □

Getting back to the proof of the theorem, apply the Δ-system lemma to the collection $\mathcal{A} = \{ F_U : U \in \mathcal{U} \}$ to find an uncountable Δ-system $\mathcal{B} \subseteq \mathcal{A}$. Let r be the root of this Δ-system.

Claim. $r \neq \emptyset$.

Proof. Suppose for the sake of contradiction that $r = \emptyset$. Given two sets $F, G \in \mathcal{B}$, let U and V be elements of $\mathcal{U}$ such that $F_U = F$ and $F_V = G$ (U and V need not be unique since the map $U \mapsto F_U$ need not be injective). Since $r = \emptyset$, this means $F \cap G = F_U \cap F_V = \emptyset$, which would in turn imply that $U \cap V \neq \emptyset$ by the lemma, contradicting the assumption that $\mathcal{U}$ consists of pairwise disjoint sets. This finishes the proof of the claim. □

Let $\mathcal{U}' = \{ U \in \mathcal{U} : F_U \in \mathcal{B} \}$. For each $U = \prod_{\alpha \in I} U_\alpha \in \mathcal{U}'$, define:

$$p(U) = \prod_{\alpha \in r} U_\alpha.$$

("p" for "projection", since p is essentially projecting each $U \in \mathcal{U}'$ to the coordinates in r.)

Finally, let $\mathcal{V} = \{ p(U) : U \in \mathcal{U}' \}$. So $\mathcal{V}$ is a collection of open subsets of $\prod_{\alpha \in r} X_\alpha$, a space which is ccc by hypothesis, since r is finite.

Claim. *The map $\mathcal{U}' \to \mathcal{V}$ given by $U \mapsto p(U)$ is injective.*

Proof. Let $U = \prod_{\alpha \in I} U_\alpha$ and $V = \prod_{\alpha \in I} V_\alpha$ be given elements of $\mathcal{U}'$, and note that by definition of $\mathcal{U}'$, we have that $F_U \cap F_V = r$.

Then if $U \neq V$ and $p(U) = p(V)$, we would have that $U_\alpha = V_\alpha$ for all $\alpha \in r = F_U \cap F_V$. By the lemma, this would imply that $U \cap V \neq \emptyset$, contradicting the assumption that $\mathcal{U}$ consists of pairwise disjoint sets. □

It follows from the last claim that $\mathcal{V}$ is uncountable, since $\mathcal{U}'$ is uncountable. By hypothesis, $\prod_{\alpha\in r} X_\alpha$ is ccc, and so in particular there must exist $p(U), p(V) \in \mathcal{V}$ such that $p(U) \cap p(V) \neq \emptyset$. But then, by the lemma again, we could find a point in X inside $U \cap V$, again contradicting the assumption that $\mathcal{U}$ consists of disjoint sets. Therefore, the whole product space X must have the countable chain condition.

Finally, we treat the case in which I is countable, which is much simpler than the previous case. Again, assume for the sake of contradiction that $\mathcal{U}$ is an uncountable collection of pairwise disjoint, nonempty, open subsets of X. For each $U \in \mathcal{U}$, let $F_U \subseteq I$ be as in the previous case. There are only countably many finite subsets of I, and therefore by the pigeonhole principle there must be a finite $F \subseteq I$ and an uncountable $\mathcal{U}' \subseteq \mathcal{U}$ such that $F_U = F$ for all $U \in \mathcal{U}'$. In other words, $\pi_\alpha(U) = X_\alpha$ for all $\alpha \in I \setminus F$ and all $U \in \mathcal{U}'$.

We are now in essentially the same situation as in the previous proof (but now with F instead of r). For each $U = \prod_{\alpha\in I} U_\alpha \in \mathcal{U}'$, let $p(U) = \prod_{\alpha\in F} U_\alpha$. Then the mapping $U \mapsto p(U)$ is injective, and therefore since $\prod_{\alpha\in F} X_\alpha$ is ccc by hypothesis, there must exist $U, V \in \mathcal{U}'$ such that $p(U) \cap p(V) \neq \emptyset$. But then, by the lemma, this implies that $U \cap V \neq \emptyset$, contradicting the assumption that $\mathcal{U}$ consists of pairwise disjoint sets. □

It follows easily from the theorem that if ϕ is a topological property such that:

- ϕ is finitely productive;
- ϕ implies the ccc,

Then any product of topological spaces with ϕ will also have the ccc. In particular:

Corollary 2.6. *Let $\{\, (X_\alpha, \mathcal{T}_\alpha) : \alpha \in I \,\}$ be a family of separable spaces. Then their product $\prod_{\alpha\in I} X_\alpha$ has the countable chain condition.*

This result allows us to easily construct ccc, non-separable spaces. For example, if I is an indexing set with cardinality strictly larger than $\mathbb{R}$, then $\{0,1\}^I$ is ccc but not separable.

Another notable corollary:

Corollary 2.7. *If, under some additional set-theoretic assumptions, we can prove that the ccc is finitely productive, we will also have shown that, under those assumptions, the ccc is arbitrarily productive.*

In the next section, we will use this corollary to prove that under a certain additional set-theoretic assumption the ccc is arbitrarily productive.

3 Martin's Axiom implies that the ccc is productive

Martin's Axiom (MA) is a statement that is independent of the usual ZFC axioms of set theory. It is a useful tool for proving many things, including some independence results. It has many forms, but by far the most useful way to think about it is as a statement about partial orders.

We have seen what a partial order is (in section 10 of the lecture notes), but we will need a few more definitions before we can see the statement of Martin's Axiom. Some of these we have seen before, and some of them are partial order versions of things we have seen in a topological context.

3.1 Preliminaries

Definition 3.1. *Let* $(\mathbb{P}, \leq)$ *be a partial order.*

- *Two elements* $p, q \in \mathbb{P}$ *are called* compatible *if there is an element* $r \in \mathbb{P}$ *such that* $r \leq p$ *and* $r \leq q$*. If they are not compatible they are called* incompatible*. A subset* $A \subseteq \mathbb{P}$ *is called an* antichain *if every pair of elements of* A *are incompatible.*

- $(\mathbb{P}, \leq)$ *is said to have the* countable chain condition *(or* ccc*) if every antichain in* $\mathbb{P}$ *is countable.*

- *A subset* $D \subseteq \mathbb{P}$ *is called* dense in $\mathbb{P}$ *(or simply* dense *if the partial order in question is clear from context) if for every* $p \in \mathbb{P}$ *there is a* $d \in D$ *such that* $d \leq p$*.*

- *A strict, non-empty subset* $\mathcal{F} \subseteq \mathbb{P}$ *is called a* filter on $\mathbb{P}$ *if the following conditions hold:*
  1. $\mathcal{F}$ *is closed upwards. In other words, if* $p \in \mathcal{F}$ *and* $p \leq q$*, then* $q \in \mathcal{F}$*.*
  2. $\mathcal{F}$ *is directed. In other words, if* $p, q \in \mathcal{F}$*, then there is a* $r \in \mathcal{F}$ *such that* $r \leq p$ *and* $r \leq q$*. Another way to say this is that every two elements of* $\mathcal{F}$ *are compatible, and that this compatibility is witnessed by an element of* $\mathcal{F}$*.*

With these definitions in place, we can state Martin's Axiom. To be a little more precise, we are going to state what is usually called MA(ω_1).

(MA($\boldsymbol{\omega_1}$)) Let $(\mathbb{P}, \leq)$ be a partial order with the countable chain condition, and let $\{ D_\alpha : \alpha \in \omega_1 \}$ be a collection of non-empty, dense subsets of $\mathbb{P}$. Then there is a filter $\mathcal{G}$ on $\mathbb{P}$ such that $\mathcal{G} \cap D_\alpha \neq \emptyset$ for all $\alpha \in \omega_1$.

This statement is independent of the usual axioms of ZFC (the proof of this is, sadly, well beyond the scope of this note and our course). In other words, if ZFC is consistent, then assuming MA(ω_1) is true cannot lead to a contradiction, and neither can assuming it is not true. It is with the help of this and many other statements like this that mathematicians can prove certain statements are consistent with ZFC.

3.2 Results

We will now prove that under the assumption that ($\mathrm{MA}(\boldsymbol{\omega_1})$) is true, any product of ccc topological spaces is ccc. We start with a preliminary lemma.

Lemma 3.2. *Assume ($MA(\boldsymbol{\omega_1})$).*

Let $(X, \mathcal{T})$ be a ccc topological space, and suppose $\{ U_\alpha : \alpha \in \omega_1 \}$ is a collection of non-empty open subsets of X. Then there is an uncountable $A \subseteq \omega_1$ such that for any $\alpha, \beta \in A$, $U_\alpha \cap U_\beta \neq \emptyset$.

Proof. For each $\alpha \in \omega_1$, define

$$V_\alpha = \bigcup_{\beta > \alpha} U_\beta.$$

These new open sets form a decreasing chain of sets, in the sense that if $\alpha < \beta$, then $V_\beta \subseteq V_\alpha$.

We first show that this chain "stabilizes" at some point. More specifically, that there is an $\alpha \in \omega_1$ such that for all $\beta > \alpha$, $\overline{V_\beta} = \overline{V_\alpha}$. Indeed, if no such α exists, then for every $\beta \in \omega_1$ we can find a $\gamma_\beta > \beta$ such that $W_\beta := V_\beta \setminus \overline{V_{\gamma_\beta}} \neq \emptyset$. Any two sufficiently far-apart sets of this form are disjoint: if $\alpha \in \omega_1$ and $\beta > \gamma_\alpha$, then $W_\alpha \cap V_{\gamma_\alpha} = \emptyset$ while $W_\beta \subseteq V_\beta \subseteq V_{\gamma_\alpha}$. Therefore, proceeding this way, we can construct an unbounded—and therefore uncountable—subset of ω_1 and a corresponding uncountable collection of disjoint open subsets of X—the associated W_α's—contradicting the assumption that X has the ccc.

So, fix an $\alpha \in \omega_1$ after which the chain of V_α's stabilizes, as in the previous paragraph. Consider the partial order $(\mathbb{P}, \subseteq)$, where

$$\mathbb{P} = \{ P \subseteq V_\alpha : P \text{ is open and non-empty} \}.$$

$\mathbb{P} \subseteq \mathcal{T}$, and so $\mathbb{P}$ has the ccc (as a partial order) since $(X, \mathcal{T})$ has the ccc (as a topological space). We want to apply $\mathrm{MA}(\omega_1)$ to $\mathbb{P}$, and so we need some dense subsets of $\mathbb{P}$ to work with.

For each $\beta \in \omega_1$, define a set $D_\beta \subseteq \mathbb{P}$ by:

$$D_\beta = \{ P \in \mathbb{P} : P \subseteq U_\gamma \text{ for some } \gamma > \beta \}.$$

Claim. *For all $\beta \in \omega_1$, D_β is dense in $\mathbb{P}$.*

Proof. Fix a $\beta \in \omega_1$. By definition of α, we have that $\overline{V_\alpha} \subseteq \overline{V_\beta}$ (if $\alpha < \beta$ then in fact we have $\overline{V_\alpha} = \overline{V_\beta}$, while if $\beta < \alpha$ then this follows from our early observation that these sets form a decreasing chain).

Fix an arbitrary $P \in \mathbb{P}$. We want to find a $Q \in D_\beta$ such that $Q \subseteq P$. From our observation just above, it follows that $P \subseteq \overline{V_\beta}$, and in particular that $P \cap V_\beta \neq \emptyset$. In other words, recalling the definition of V_β

$$P \cap \left(\bigcup_{\gamma > \beta} U_\gamma \right) \neq \emptyset,$$

whence it follows that $P \cap U_\gamma \neq \emptyset$ for some $\gamma > \beta$. Let $Q = P \cap U_\gamma$. Then, to summarize:

- Q is open, and $Q \subseteq P$.
- $Q \subseteq U_\gamma$ and $\gamma > \beta$, and so $Q \in D_\beta$.

This completes the proof of the claim. □

Finally, we are in a position to apply (MA(ω_1)). Indeed, let $\mathcal{G} \subseteq \mathbb{P}$ be a filter such that

$$\mathcal{G} \cap D_\beta \neq \emptyset \quad \text{for all} \quad \beta \in \omega_1.$$

We will define the subset $A \subseteq \omega_1$ that we are looking for using $\mathcal{G}$ in the following way:

$$A := \{ \gamma \in \omega_1 \,:\, P \subseteq U_\gamma \text{ for some } P \in \mathcal{G} \}.$$

Claim. *A is uncountable.*

Proof. We will show that A is unbounded in ω_1, from which it follows that it is uncountable. Indeed, fix an arbitrary $\beta \in \omega_1$. By construction, $\mathcal{G} \cap D_\beta \neq \emptyset$, so let P be an element of this intersection. By definition of D_β, this means there is a $\gamma > \beta$ such that $P \subseteq U_\gamma$. But then $\gamma \in A$, completing the proof of the claim. □

Claim. *For any $\alpha, \beta \in A$, $U_\alpha \cap U_\beta \neq \emptyset$.*

Proof. This follows immediately from the fact that $\mathcal{G}$ is a filter, and is therefore directed. Indeed, let $\alpha, \beta \in A$. By definition of A, this means there are $P_\alpha, P_\beta \in \mathcal{G}$ such that $P_\alpha \subseteq U_\alpha$ and $P_\beta \subseteq U_\beta$. But $\mathcal{G}$ is a filter, and so $P_\alpha \cap P_\beta \neq \emptyset$, from which it follows that $U_\alpha \cap U_\beta \neq \emptyset$. □

□

The preceding claim is a bit technical, but having proved it we can now get the result we want very easily.

Theorem 3.3. *Assume (MA(ω_1)).*
Then any product of ccc topological spaces is ccc.

Proof. By Corollary 2.6, it suffices to show that the product of any two ccc spaces is ccc.

Let X and Y be ccc topological spaces, and assume for the sake of contradiction that $\{ W_\alpha \,:\, \alpha \in \omega_1 \}$ is a collection of pairwise disjoint, non-empty open subsets of $X \times Y$. Shrinking each one if necessary, we may assume without loss of generality that $W_\alpha = U_\alpha \times V_\alpha$, where $U_\alpha \subseteq X$ and $V_\alpha \subseteq Y$ are open sets.

Applying the lemma to the collection $\{ U_\alpha \,:\, \alpha \in \omega_1 \}$, we find an uncountable $A \subseteq \omega_1$ such that for any $\alpha, \beta \in A$, $U_\alpha \cap U_\beta \neq \emptyset$. Fix arbitrary $\alpha \neq \beta \in A$. Then $U_\alpha \cap U_\beta \neq \emptyset$ as we just said, but on the other hand $(U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = W_\alpha \cap W_\beta$ is empty by assumption. This forces $V_\alpha \cap V_\beta = \emptyset$. Since α and β were arbitrary, this means $\{ V_\alpha \,:\, \alpha \in \omega_1 \}$ is a pairwise disjoint collection of non-empty open subsets of Y, contradicting the assumption that Y has the ccc. □