MAT382: HOMEWORK SET #4

DUE: MONDAY MARCH 12, 2018

You are expected to write complete proofs to the following problems - leaps of logic or unproven assumptions will be penalized. You may use any statement we've proven in class freely. Please ask me if any question is unclear. Every question is worth 10 points, making the homework out of 40.

(1) For a Dirichlet series $D(s) = \sum_n \frac{a_n}{n^s}$, define $D'(s) = \sum_n \frac{-a_n \log n}{n^s}$. Prove that if $F(s) = D(s)E(s)$, then $F'(s) = D'(s)E(s) + D(s)E'(s)$.

(2) For a Dirichlet series $D(s) = \sum_n \frac{a_n}{n^s}$, we define $D(ms+k) = \sum_n \frac{a_n n^{-k}}{(n^m)^s}$, which we often write shorthand as

$$D(ms+k) = \sum_n \frac{a_n n^{-k}}{n^{ms}}.$$

For a positive integer $k \geq 2$, let

$$F_k(n) = \begin{cases} 1 & n \text{ is not divisible by a } k\text{'th power of an integer greater than 1} \\ 0 & \text{else} \end{cases}$$

so for $k = 2, F_k(n)$ simply determines whether an integer is square-free. Prove that $\sum_n \frac{F_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$

(3) Recall that $d(n)$ is equal to the number of positive integers dividing n. Prove that

$$\sum_n \frac{d(n)}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}.$$

Conclude that $\mu^2 * d = \mu * d^2$ where f^2 denotes the function $f^2(n) = f(n)^2$.

(4) Let $F(n)$ be the number of triples of positive integers (a, b, c) such that $\gcd(a, b, c) = 1$ and $\max(a, b, c) \leq n$. Compute $\lim_{n\to\infty} \frac{F(n)}{n}$.

(5) (Bonus - 1% on the course) Let $f(n)$ be the function which is 1 if n is the fourth power of a prime, and 0 otherwise. Let $f, g : \mathbb{Z}_+ \to \mathbb{Z}$ be arithmetic functions such that $f = g * h$. Prove that $g(1) = \pm 1$ or $h(1) = \pm 1$.