WEEK 8: PLAYING GAMES

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Games

Two player games are very common. A *winning strategy* for one of the two players (Alice) is a set of rules to follow, such that no matter what the other player does, if Alice follows the rules she will win the game. A *winning position* for one of the players is a starting configuration for which that player has a winning strategy. It is a fact that if Alice and Bob play a game which ends in a finite amount of time, and one of the two players always wins, then there is a winning strategy for either Alice or Bob.[1]

Most games we play are too complicated to be solved (or else they wouldn't be much fun!) but even so, they contain little subgames which can - and often are - analyzed completely. For example, think of end games in chess. Here are some tricks and tips for analyzing games:

(1) Use induction! Try small cases, and spot a pattern. This is always true, but even more so in games. If a game is played with 100 checkers with both players removing checkers according to some rule, see what happens for 1 or 2 or 3 checkers first!
(2) Notice if the game is symmetric for the two players. If not, careful how you set up your induction!
(3) Try to find winning and losing positions. Even if they don't include the specific starting configuration you care about, you can try to find a pattern and then prove it using induction or some other method.

[1] This is by no means a trivial remark!

(4) Look for invariants! Often a game seems complicated but some much simpler thing is going on, and you can keep track of it by finding the appropriate invariant. For example, if Alice can always leave an odd number of matchsticks, she never has to worry about taking the last one!
(5) Remember that in a 2 player game, a winning position for one of the players is any position that can move to a losing position for the other player. A losing position is a position such that every move makes it a winning position for the other player.

2.1. **Starter Problems.** Here are some problems to get you started:

(1) (The matchstick game) Alice and Bob play a game with a pile of matches, with Alice moving first. On each players turn, they must remove 1 or 2 matches from the pile. The person who empties the pile wins.The initial pile has 100 matches. Figure out a winning strategy for one of the players.
(2) (Misère matchsticks)Alice and Bob play the same game as above, only now the player who empties the pile *loses*. Figure out a winning strategy for one of the players.
(3) There is a table with a square top of radius 10. Two players take turn putting a dollar coin of radius 1 on the table. The player who cannot do so loses the game. Show that the first player can always win.
(4) Alice and Bob play a game in which the first player places a king on an empty 8×8 chessboard, and then, starting with the second player, they alternate moving the king (in accord with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?
(5) Alice and Bob play the following game on a 100×100 grid.. There is initially a checker on square $(36, 47)$. Starting with Alice, the players may move that checker arbitrarily far down, or arbitrarily far left on their turn. The winner is the first player to get to the bottom left square $(1, 1)$. Determine who has a winning strategy.

2.2. **Followup Problems.**

(1) Alice and Bob play a game with a pile of matches, with Alice moving first. On each players turn, they must remove 1 or 2 or 3 matches from the pile. The person who empties the pile wins.The initial pile has 100 matches. Figure out a winning strategy for one of the players.
(2) (Misère matchsticks) Alice and Bob play the same game as above, only now the player who empties the pile *loses*. Figure out a winning strategy for one of the players.
(3) Alice and Bob play a game in which the first player places a king on an empty 5×5 chessboard, and then, starting with the second

player, they alternate moving the king (in accord with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?

(4) Alice and Bob play the following game on a 100×100 grid. There is initially a checker on square $(36, 47)$. Starting with Alice, the players may move that checker arbitrarily far down, or arbitrarily far left on their turn. The *loser* is the first player to get to the bottom left square $(1, 1)$. Determine who has a winning strategy.

3. More Problems

(1) Alice and Bob again play the matchstick game only now there are 100 matches, and the players may remove 2^m matches for any non-negative integer m (so you may remove $1, 2, 4, 8, \dots$ matches). Figure out a winning strategy for one of the players (last player to empty the pile wins).

(2) (harder) Alice and Bob play the matchstick game again, but now there are n matches in the pile, and they are allowed to remove $1, 3$ or 8 matches at a time. Alice plays first and the person to empty the pile wins. For which n does Alice have a winning strategy?

(3) Alice and Bob play a game as follows. They start with a row of 50 coins, of various values. The players alternate, and at each step they pick either the first or last coin and take it. The winner is the one with more money at the end. If Alice plays first, prove that she has a winning strategy. Find an example where Bob can make more money than Alice if there are 51 coins.

(4) Same question(s) as 2.1#3 and 2.2#3, only now with a knight instead of a king.

(5) Alice and Bob alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent (i.e. not a side) and if the diagonal formed does not cross any of the previous diagonals formed. The last player to draw a diagonal wins. Who has a winning strategy if the polygon has 90 sides and Alice moves first?

(6) Let n be a positive integer. Alice and Bob play a game with a set of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of the players. Beginning with Alice, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$, and the last player to discard wins the game. Prove that Bob has a winning strategy.

(7) The game of Chomp is played on an $m \times n$ board by Alice and Bob as follows. Alice moves first. On a players move, they must place an X on any square (i, j) which does not yet have an X on it, and they

also place an X on any square above and to the right of that square which does not yet have an X on it. That is, any square (s,t) with $s \geq i$ and $t \geq j$ which does not yet have an X also gets an X put in it. The person who places the last X loses. Determine a winning strategy on a 3×3 game of chomp. What about on a 100×100 game of chomp? Now figure out who wins on a 100×101 game of chomp *hint, it is unknown what a winning strategy is!*

(8) Two players play a game on a 3×3 board. The first player places a 1 on an empty square and the second player places a 0 on an empty square. Play continues until all squares are occupied. The second player wins if the resulting determinant is 0 and the first player wins if it has any other value. Who wins?

(9) A polyhedron has at least 5 faces, and it has exactly 3 edges at each vertex. Two players play a game. Each in turn selects a face not previously selected. The winner is the first to get three faces with a common vertex. Show that the first player can always win.