HOMEWORK SET #4: DUE NOVEMBER 12

(1) Prove that for $n \geq 1$, we have $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$.

(2) Suppose n disks, black on one side and white on the other, are laid out in a straight line with a random arrangement of black sides up. You are playing a game of solitaire, in which a turn consists of removing a black disk and flipping over its immediate neighbors, if any. (Two disks are not considered immediate neighbors if there used to be a disk between them that is now gone.) A game is winnable if it is possible to remove all n disks. Prove that a game is winnable if and only if the number of disks that are black side up is odd.

(3) Two players are playing a game with raisins. They start with two non-empty piles of raisins on the table, and take turns making moves. A move consists of eating all the raisins in one pile and dividing the second into two non-empty (and not necessarily equal) piles. The game continues until one of the players can't move; that player loses. Show that if one of the piles starts out with an even number of raisins then the first player can win.

(4) How many 100 digit numbers are there with no pair of equal consecutive digits? (355 has a pair of consecutive digits but 353 does not).

(5) Prove that $\binom{2n}{n} = 2 \cdot \binom{2n-1}{n}$.

(6) In a 8×8 chessboard some of the cells are colored white and the rest are colored black. Such a coloring is called *cool* if in every row and column, and odd number of cells are black. Prove that the number of cool colorings is 2^{49}.