MAT 475 WEEK 6: INDUCTION

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Induction

If you want to prove some family of statements ϕ_n, it is enough to prove ϕ_1, and show that ϕ_m implies ϕ_{m+1} for all m. A variant is **strong induction**: Instead of showing ϕ_m implies ϕ_{m+1}, it is enough to prove that $\phi_1, \phi_2, \ldots, \phi_m$ take altogether imply ϕ_{m+1}. This can be very helpful in some applications

Here are some problems to get you started:

(1) Prove any integer $n \geq 4$ can be written as $n = 2a + 5b$ for positive integers a, b.
(2) Define the Fiobonacci numbers by $F_1 = F_2 = 1$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Prove that for all n, $\gcd(F_n, F_{n+1}) = 1$.
(3) Consider the sequence of real numbers $x_1 = 1, x_{n+1} = \sqrt{1 + 2x_n}$. Prove that $x_n < 4$ for all n.
(4) A country has n cities. Any two cities are connected by a one-way road. Show that there is a route that passes through every city.

2.1. Followups to the above.

(1) Prove any integer $n \geq 24$ can be written as $n = 5a + 7b$ for positve integers a, b.
(2) Prove that $\gcd(F_n, F_{n+2}) = 1$ for all n.

2.2. Pigeonhole Problems.

(1) Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
(2) Find all positive integers n such that $n! > 3^n$.
(3) Prove that $1 + 2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
(4) Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.
(5) Let $a_1 < a_2 < \cdots < a_n$ be a sequence of positive real numbers, and let $b_1, \ldots, b_n$ be the same numbers in a possibly different order (in other words, a permutation). Suppose

$$a_1 + b_1 < a_2 + b_2 < \cdots < a_n + b_n.$$

Prove that $a_i = b_i$ for all i.
(6) One square is removed from a $2^n \times 2^n$ board. Prove that the remaining squares can be tiled by L-shaped triminoes.

(7) Consider all non-empty possible subsets of $\{1, 2, \ldots, N\}$ which contain no 2 consecutive integers. For each such subset, multiply all its elements. Prove that the sums of the squares of all of these numbers is $(N+1)! - 1$.*As an example, for* $N = 4$ *the subsets are* $\{1\}, \{2\}, \{3\}, \{1, 3\}$ *with products* $1, 2, 3, 3$. *The squares of these products are* $1, 4, 9, 9$ *and the sum is* $23 = 4! - 1$.

(8) Let $\{a_i\}$ be a sequence of real numbers satisfying $a_i + a_j \leq a_{i+j}$ for all i, j. Prove that

$$\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \leq a_n.$$

Hint: This is definitely strong induction!