MAT 475 WEEK 4: PIGEONHOLE

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Pigeonhole Principle

The Idea is very simple: if you have $kn+1$ pigeons living in n holes, then some hole contains at least $k+1$ pigeons. In practise, this can be quite hard to implement! The reason this can be tricky to apply is that one can get very creative with what are the 'pigeons' and what are the 'holes'.

Here are some problems to get you started:

(1) Consider 5 points inside a square of radius 2. Prove 2 of the points are at most $\sqrt{2}$ apart.
(2) Prove that in any group of 6 people there are either 3 mutual friends or 3 mutual strangers. *friendship is symmetric, so if A is friends with B then B is friends with A.*
(3) Prove that any 7×3 rectangle, each square of which is colored red or blue, contains a subrectangle whose vertices are all the same color. Is the same thing true for a 6×3 rectangle?
(4) A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

2.1. Followups to the above.

(1) Consider 7 points inside a 3×4 rectangle. Prove 2 of the points are at most $\sqrt{5}$ apart.
(2) Prove that any 19×4 rectangle, each square of which is colored red or blue or green, contains a subrectangle whose vertices are all the same color.
(3) $a_1, \ldots, a_{100}$ are positive integers that add to at most 150. Prove that there are positive integers $1 < m \leq n \leq 100$ such that $\sum_{k=m}^{n} a_k = 25$.

2.2. Pigeonhole Problems.

(1) Five lattice points are chosen in the plane lattice. Prove that you can choose two of these points such that the segment joining these points passes through another lattice point. (The plane lattice consists of all points of the plane with integral coordinates.)
(2) Choose 9 positive integers between 1 and 200. Prove that 2 of these integers have a ratio that lies between 1 and 2.
(3) Let N be a positive integer. Prove that some multiple of N has a decimal expansion consisting of 0's and 1's.

(4) There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.
(5) Show that the decimal representation of any rational number p/q must eventually repeat.
(6) Seventeen people are all mutual friends. Each pair of people does of the following three activities together on a regular basis: Judo, Squash, and Jogging. Prove that for some 3 of these people, the 3 activities that the 3 different pairs participate in are all the same.
(7) $a_1, \ldots, a_{1000}$ are positive integers. Prove that there exists i, j with $1 \leq i \leq j \leq 1000$ such that $\displaystyle\sum_{k=i}^{j} a_k$ is divisible by 1000.
(8) Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.
(9) 101 people with distinct heights are lined up in a row. Show that we can pick 11 of them, such that their heights are either increasing or decreasing along the sequence.
(10) Let T be the set of positive integer divisors of 60^{100}. What is the e largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

2.3. **Hints.**

(1) Consider the midpoints.
(2) Break [1,200] up into [1,2], [2,4], [4,8] ...
(3) Consider the residues of the powers of 10 modulo N.
(4) Break the triangle up into 4 smaller shapes.
(5) What is the decimal representation of $10p/q$?
(6) Pick a person. They must do some activity with at least 6 other people. Now focus on those 6.
(7) Consider all the partial sums $a_1 + a_2 + \cdots + a_j$ for $1 \leq j \leq 1000$.
(8) Pick a bunch of points on a horizontal line of one color. Now what has to be true on every other horizontal line?
(9) Label each person with the length of the longest decreasing sequence starting at that person, as well as the longest increasing sequence starting at that person.
(10) This is really about points in a $200 \times 100 \times 100$ cube.