MAT 475 WEEK 11: POLYNOMIALS

JACOB TSIMERMAN

These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Polynomials

2.1. **notation.** A *monomial* is an expression of the form ax^k for some positive integer k and some number a. A *polynomial* $P(x)$ is the sum of finitely many monomials. We say that $P(x)$ has *degree* n if we can write $P(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n \neq 0$. Polynomials of degree 0 are called constants, of degree 1 are called linear polynomials, and of degree 2 are called quadratic polynomials. The numbers a_i are called the *coefficients* of $P(x)$. We say a polynomial is an *integer, rational, real, or complex* polynomial if the coefficients are. If $P(x)$ is degree n, we say $a_n x^n$ is the *leading term* and a_n is the leading coefficient.

2.2. **Roots.** A (integer, real, or complex) number r is called a *root* of $P(x)$ if $P(r) = 0$. This is equivalent to being able to write $P(x) = (x - r)Q(x)$ for some other polynomial $Q(x)$ of degree 1 less then the degree of $P(x)$. It follows that a non-zero polynomial of degree n has at most n roots.

It is in fact true that any polynomial over the complex numbers can be factored into n linear factors. I.e. Every complex polynomial can be uniquely written as

$$P(x) = a_n \prod_{i=1}^{n} (x - r_i)$$

for complex numbers $r_1, \ldots, r_n$.

This means we can find all the coefficients of the polynomial in terms of the leading coefficient and the roots. For example, for a quadratic polynomial

$P(x) = ax^2 + bx + c$ with roots r_1, r_2 we have

$$ax^2 + bx + c = a(x - r_1)(x - r_2) = a(x^2 - (r_1 + r_2)x + r_1 r_2$$

which means that $b = ar_1 + ar_2, c = ar_1 r_2$.

2.3. **Tricks.** Some tricks to look for:

- To show 2 polynomials of degree n are equal, all you have to do is show they are equal at $n + 1$ different points.
- Use induction on the degree of the polynomials.
- Factor: If you know a root r of $P(x)$, write $P(x) = (x - r)Q(x)$.
- A non-zero polynomial $P(x)$ is degree $n \geq 1$ if and only $P(x + 1) - P(x)$ is of degree $n - 1$.
- Write the coefficients in terms of the roots. Even though you often can't find the roots exactly, you know that the coefficients equal certain symmetric expressions in the roots.

2.4. **Started Problems.** Here are some problems to get you started:

(1) Suppose that $f : \mathbb{Z} \to \mathbb{R}$ is a function from the integers to the reals satisfying $f(n + 2) - 2f(n + 1) + f(n) = 2$ for all integers n. Prove that f equal to a quadratic polynomial with leading coefficient 1.
(2) Prove that if $P(x)$ is an integer polynomial then for any integers $m \neq n$ we have $(m - n)|(P(m) - P(n))$.
(3) Let $fPx) = x^{10} - 2x^9 + 1$, and let $Q(x) = P(x + 1)$. Note that $Q(x)$ is also a polynomial. What is the sum of the coefficients of $Q(x)$?
(4) What is the sum of the squares of the roots of $P(x) = x^3 - 10x^2 + 17x - 5$? You may assume all the roots are distinct.

2.5. **Problems to follow up on.** :

(1) Find numbers a, b, c, d such that if $f : \mathbb{Z} \to \mathbb{R}$ satisfies $af(n + 3) + bf(n + 2) + cf(n + 1) + df(n) = 1$ then f is equal to a degree 3 polynomial with leading coefficient 1.
(2) Find an integer polynomial whose coefficients are not divisible by 3, but its value at any positive integer is divisible by 3.
(3) What is the sum of the reciprocals of the roots of $P(x) = x^3 - 10x^2 + 17x - 5$? You may assume all the roots are distinct.

3. Polynomial Problems

These are in increasing order of difficulty. The ones at the end are extremely challenging!

(1) Suppose that $P(x)$ is a degree n polynomial satisfying $P(i) = 0$ for $i = 0, 1, \ldots, n - 1$ and $P(n) = 1$. Determine $P(n + 1)$.
(2) Suppose that $P(x)$ is a degree n polynomial satisfying $P(i) = i$ for $i = 0, 1, \ldots, n - 1$ and $P(n) = n + 1$. Determine $P(n + 1)$.
(3) Let a, b, c be distinct, positive integers. Prove there does not exist a polynomial $P(x)$ with integer coefficients satisfying $P(a) = b, P(b) = c, P(c) = a$.

(4) A polynomial $P(x)$ of degree n satisfies $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \dots, n$. Find $P(n+1)$.
(5) Let $P(x)$ be a real polynomial which takes integer values whenever x is an integer. Prove that $n!P(x)$ is an integer polynomial.
(6) $P_1(x) = x^2 - 2$, and define $P_n(x)$ recursively via $P_{n+1}(x) = P_1(P_n(x))$. Prove that all the roots of $P_n(x)$ are real and distinct.
(7) Determine all polynomials $P(x)$ satisfying $P(x^2) = P(x)^2$.
(8) (a) Let $P(x)$ be a degree n polynomial such that for each integers i satisfying $0 \leq i \leq n$, $P(i)$ has the same sign as $(-1)^i$. Prove that $P(x)$ has n distinct, real roots.
    (b) Let $P(x)$ be a real polynomial of degree n. Prove that you can write $P(x) = R(x) + S(x)$ for real, degree n polynomials $R(x), S(x)$ each of which has n distinct, real roots.
(9) Does there exist a finite set S of non-zero real numbers such that there exist polynomials of arbitrarily high degree, whose roots and coefficients are all in S? *Harder: What if S is allowed to be a finite set of non-zero complex numbers?*