MAT 495 WEEK 9: PROBABILITY

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Probability

When approaching probability problems, the most important fact to remember is that if two events X and Y are independent, then the probabilities that they both occur is the product of the probabilities that each of them occurs. Formally, $P(XY) = P(X)P(Y)$.

When approaching probability problems, here are some hints:

(1) Look for symmetry. Even though an event might be incredibly complicated to fully understand, the presence of symmetry often yields that process largely irrelevant. For example, if m, n are two positive integers which are produced using the *same* probabilistic method, then $P(m > n) = P(n > m)$.
(2) Isolate the parts of the random process that are relevant to you. For example, If I flip a random coin 500 times, the odds that it ends in (tails, heads) only depends on the last two flips!

2.1. **Starter Problems.** Here are some problems to get you started:

(1) I've just heard a public announcement that 80% of UToronto students don't drink (to oblivion), 70% don't get lazy(on a regular basis), and 51% don't party (every night). Prove mathematically that at least one UToronto student came here to study.
(2) A fair coin (one which lands on heads half the time, and on tails half the time) is tossed 3 times. Prove that it is 50% likely to land heads an odd number of times.

(3) Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?
(4) One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of the unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in her assigned seat?
(5) three points A, B, C are chosen uniformly at random on a circle. What is the probability that the center of the circle lies in the triangle ABC?

2.2. **Followup Problems.**

(1) A stores sells apples. The chance of a randomly picked apple from the store being rotten 1/3, the chance of it having a worm is 1/2, the chance of it being soggy is 1/4, and the chance of it being rotten AND soggy is 1/10. Prove that this store has at least one apple that has no worms, and is neither soggy nor rotten.
(2) Two fair coins (one which lands on heads half the time, and on tails half the time) are tossed 3 times each. Prove that it is 50% likely that exactly one of the coins landed heads an odd number of times.
(3) Slips of paper with the numbers from 1 to 99 are placed in a hat. Three numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the second number to be drawn is smaller then both the first and third?

3. Probability Problems

(1) The buses in Toronto going to the latest Blue Jays game were really full. Any bus that had 20 or more people riding in it was considered overfilled. Prove that the percentage of overfilled buses was at most as large as the percentage of passengers that were riding an overfilled bus.
(2) Indra has 5 pairs of shoes, each of a different style. If she selects two shoes at random, what is the probability that she will select a matching pair? What is the probability that she will select 1 right shoe and 1 left shoe?
(3) Two baseball teams are playing each other in a best of 7 match, with each game having a definite winner (no draws). The teams are equally matched so that each game has an equal probability of either team winning. Prove that the probability that the game will end in 6 games is the same as the probability that the game will end in 7 games. *You do not have to compute the probabilities themselvesl!*

Hint: If there are at least 6 games, what score HAS to occur during the match?

(4) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

(5) You have $n > 1$ numbers $0, 1, \ldots, n-1$ arranged on a circle. Johnny starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that, when the walker is at i for the first time, all other points have been previously visited, i.e., that i is the last new point. For example, $p_0 = 0$. **Hint: Try small cases!!!**

(6) Choose two random numbers from the closed interval $[0, 1]$ and let them be the endpoints of an interval. Repeat this n times, so that you end up with n intervals, which may or may not intersect. What is the probability that one of the intervals contains all the others?

(7) Three players are in a room and a red or blue hat is placed on each person's head. The color of each hat is determined by [an independent] coin toss. No communication of any sort is allowed, but the players are allowed an initial strategy session to decide on a mutual strategy before the game begins. Once they have had a chance to look at the other hats *but not their own*, the players must simultaneously guess the color of their own hats or pass. The players win if no-one guesses incorrectly, and at least one person does not pass. Find a strategy that maximizes their odds of winning.

(8) Let p_n be the probability that $c + d$ is a perfect square, where the integers c, d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Find the limit of $p_n\sqrt{n}$ as n tends to infinity.