WEEK 7: GRAPH THEORY

JACOB TSIMERMAN

These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Graphs: Some notation

A graph is a pair of sets (V, E), where elements of E consists of pairs of elements of V. The elements of V are called as *vertices* and the elements of E are known as *edges*. For example, the following is the graph of a triangle whose vertices are labeled $A, B, C : V = \{A, B, C]\}, E = \{AB, BC, AC\}$. Given an edge, the two elements it contains are called its *endpoints*. For example, the endpoints of the edge AB are A and B. We say the *degree* of a vertex v is the number of edges E which have v as an endpoint. So for example, in out triangle graph above, each of A, B, C has degree 2.

A *path* between two vertices v, w is a sequence of distinct vertices $v, v_1, v_2, \ldots, v_r, w$ such that every two consecutive vertices share an edge. The length of the path is $r + 1$, the number of edges. We say that G is a *connected* graph if there is a path between any two distinct vertices.

A *cycle* is a sequence of vertices $v_1, v_2, \ldots, v_r, r \geq 3$ such that every two consecutive vertices share an edge, and so do the first and last vertex.

Here are some problems to get you started:

- Suppose G is a graph with n vertices of degrees $d_1, d_2, \ldots, d_n$. If m is the number of edges that G has, prove that

$$\sum_{i=1}^{n} d_i = 2m.$$

- A *tree* is a connected graph with no cycles. Prove that a tree with n vertices has $n-1$ edges. Also, prove that a connected graph with n vertices and $n-1$ edges is a tree.
- G is a graph with 6 vertices. Prove that there are either 3 vertices in G, every two of which shares an edge, or 3 vertices in G such that no two of them share an edge.
- S is a sequence of 0's and 1's arranged in a circular pattern. We say that S is n-terrific if every subsequence of length n appears exactly once. For example, 01 is 1-terrific and 0011 is 2-terrific. Prove that if S is n-terrific, it contains exactly 2^n digits. Construct a 3-terrific set. Can you construct a 4-terrific set? 5-terrific?

3. Problems

(1) 18 soccer teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 9 teams, so that no two of them have yet played each other.

(2) Suppose that G is a connected graph. Prove that every two paths of maximum length must share at least one vertex.

(3) The *chromatic number* of a graph is the smallest number of colors K such that I can color each vertex one of K different colors such that no two vertices of the same color share an edge. Suppose G is a graph such that every vertex has degree 5. Prove that the chromatic number of G is less than 7.

(4) (Followup to the previous problem) A graph G is called bipartite if it has chromatic number at most 2. Prove that G is bipartite if and only if every cycle in G has odd length.

(5) Six points are in the plane, and between every pair either a red or a blue segment is drawn. Prove that there is a cycle of length 4, all of whose edges are the same color.

(6) There are 2000 towns in a country, each pair of which is linked by a road. The Ministry of Reconstruction proposed all of the possible assignments of one-way traffic to each road. The Ministry of Transportation rejected each assignment that did not allow travel from any town to any other town. Prove that more of half of the assignments remained.

(7) Start with a graph consisting of 10 vertices, and an edge between every pair of vertices. Now you are allowed to repeatedly do the following: find a cycle of length 4, and delete one of the edges.
   (a) Prove that you cannot end up with less than 9 edges.
   (b) Prove you can end up with exactly 10 edges.
   (c) Prove that you cannot end up with less than 10 edges.

(8) At a party, there are an even number of people, and each person has an even number of friends at the party. Prove that there are two people at the party whose number of common friends is even.

(9) A group of n people at a party has the property that each pair of persons is classified as friends or strangers. The following properties are also satisfied: Nobody is friends with everyone else, every pair of strangers has exactly one common friend, and no three people are mutually friends. Prove that everybody has the same number of friends.