CANTOR SETS

JACOPO DE SIMOI

Abstract. Some thoughts on the Cantor Set

Let us first describe how to construct the usual *middle-third* Cantor Set: we start from a set $K_0 = [0, 1]$; we remove the middle open third of this interval, that is the interval $(1/3, 2/3)$. We are left with two closed intervals which form

$$K_1 = [0, 1/3] \cup [2/3, 1]$$

. In order to construct K_2, we do the same treatment to each of the two intervals of K_1; we obtain

$$K_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

At step j, we take each of the 2^j intervals which make up K_j, remove their middle third; the union of the resulting intervals will form K_{j+1}. We then let $K = \bigcap K_j$ be the Cantor Set.

In general, for any $\alpha \in (0, 1)$ we can construct a *middle α Cantor set* by removing the middle α portion of the intervals obtained at the previous step. I claimed that whatever α we chose the measure of K would still be zero. In fact, consider the *gaps* of each approximation of the Cantor set; let ℓ_j denote the length of each interval at the j-th step and g_j be the length of the gaps introduced at step j. At step j we have 2^j intervals and thus we create 2^j gaps of length $\alpha\ell_j$ leaving us with 2^{j+1} intervals of size $(1-\alpha)\ell_j/2$, We conclude that $\ell_j = ((1-\alpha)/2)^j$. Since $g_j = \alpha\ell_j$ we have that the total measure of the gaps introduced at or before step j is:

$$m([0, 1] \setminus K_{j+1}) = 1 - \alpha \sum_{i=1}^{j} 2^j \left[\frac{1-\alpha}{2}\right]^j$$

sending $j \to \infty$ we obtain $m([0, 1] \setminus K) = 0$,

1. Dense Cantor Sets

The Cantor set has zero measure, but it is uncountable; it has also lots of gaps; what happens if we plug in smaller copies of the Cantor set in each of the gaps? And then in each of the gaps of the object that we obtain? and so on and so forth? What is the measure of the resulting object?

THINK!

2. Fat Cantor Sets

The Cantor set has zero measure, but we can easily fix that. Rather than removing always a prescribed amount α, try removing at each step j the amount α_j where $a_j \to 0$ sufficiently fast. Can you tame those gaps? Can you prescribe a sequence α_j so that the measure of the resulting Cantor set has measure ρ for any $\rho \in (0, 1)$? Observe that the gaps will still form an open and dense set.

THINK MORE!

3. Dense Fat Cantor Set

Can you combine the two constructions above? Filling in each gap of a fat Cantor another fat Cantor? Can you adjust the construction so that in the end you get something that is not full measure?

KEEP THINKING!