MAT1000HF Fall 2017
Midterm Practice Problems
Due Oct 30th

Problem 1 (Diophantine condition)
For $\gamma, \tau > 0$, define the set

$$\mathrm{DC}(\gamma,\tau) = \left\{\alpha \in \mathbb{R} \text{ s.t. } \left|\frac{p}{q} - \alpha\right| > \frac{\gamma}{|q|^{2+\tau}}\ \forall p, q \in \mathbb{Z}, q \neq 0\right\}$$
$$\mathrm{DC}(\tau) = \bigcup_{\gamma>0} \mathrm{DC}(\gamma,\tau)$$
$$\mathrm{DC} = \bigcup_{\tau>0} \mathrm{DC}(\tau)$$

Elements of this set are said to satisfy a *Diophantine condition*: they are *badly approximable with rational numbers*. Prove that:

(a) $\mathrm{DC}(\gamma,\tau) \cap \mathbb{Q} = \emptyset$ for any $\gamma, \tau > 0$.
(b) $\mathrm{DC}(\gamma,\tau)$ and $\mathrm{DC}(\tau)$ are Borel measurable.
(c) $\mathbb{R} \setminus \mathrm{DC}(\tau)$ is a Lebesgue-null set.
(d) (Hard!) There exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ so that $\alpha \notin \mathrm{DC}$.

Problem 2
Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ be a family of subsets of X. Let $\langle\mathcal{E}\rangle$ denote the σ-algebra generated by $\mathcal{E}$. Show that

$$\langle\langle\mathcal{E}\rangle\rangle = \langle\mathcal{E}\rangle$$

Problem 3
Show that if $\int_E f = 0$ for any measurable set E, then $f = 0$ almost everywhere.

Problem 4
Find an example of a sequence $f_n : \mathbb{R} \to \mathbb{R}$ that converges *uniformly* to on $\mathbb{R}$ to some function f but not in L^1. [Recall that $f_n \to f$ uniformly on a set E if for any $\varepsilon > 0$ there exists $\bar{n}$ so that for any $x \in E$ we have $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq \bar{n}$.]

Problem 5
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semi-continuous* if for any $x_0 \in \mathbb{R}$ we have $\limsup_{x\to x_0} f(x) \leq f(x_0)$. Show that every upper semi-continuous function is Borel-measurable.

Problem 6
Consider a function $F : \mathbb{R} \to \mathbb{R}$ that is non-decreasing, right continuous and

constant on the interval $[0, 1]$; let μ_F be the associated Lebesgue–Stieltjes measure and m be the Lebesgue measure on $\mathbb{R}$. Show that there exists a Lebsegue non-measurable set that is μ_F-measurable.