MAT347Y1 HW19 Marking Scheme

Friday, April 1

Total: 37 points.

14.5.5: 4 points.

14.5.8: 12 points.

(a) 6 points. Once you have two (2 points each - use minimal polynomials. Some of you tried to prove the result by giving an explicit basis for the extension, but failed to prove linear independence), the other two follow by Tower theorem (1 point each).

(b) 2 points.

(c) 4 points. Don't forget the base case, α_0 or α_1, as well as the explicit formula for $\zeta_{2^{n+2}}$ (you probably have an explicit formula for α_n, so you just need a formula for $\zeta_{2^{n+2}}$ in terms of α_n).

14.5.10: 3 points. Note that $\text{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{1\}$ IS abelian. You need to either show that $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is contained in the cyclotomic field, or prove that every intermediate extension is Galois.

14.6.13(a): 6 points.

- (1) $\alpha^2 \in \mathbb{Q} \Rightarrow \beta^2 \in \mathbb{Q}$ and $f(x) = (x^2 - \alpha^2)(x^2 - \beta^2)$
- (1) $\alpha + \beta \in \mathbb{Q} \Rightarrow \alpha\beta \in \mathbb{Q}$ and $f(x) = (x^2 + (\alpha + \beta)x + \alpha\beta)(x^2 - (\alpha + \beta)x + \alpha\beta)$
- (1) $\alpha - \beta \in \mathbb{Q} \Rightarrow \alpha\beta \in \mathbb{Q}$ and $f(x) = (x^2 + (\alpha - \beta)x - \alpha\beta)(x^2 - (\alpha - \beta)x - \alpha\beta)$
- (1) $f(x)$ has a linear factor $\Rightarrow \alpha^2 \in \mathbb{Q}$
- (2) $f(x)$ has two quadratic factors $\Rightarrow \alpha^2$, $\alpha + \beta$, or $\alpha - \beta \in \mathbb{Q}$

The contrapositive of "$f(x)$ irreducible implies $\alpha^2, \alpha \pm \beta \notin \mathbb{Q}$" is "If α^2 OR $\alpha + \beta$ OR $\alpha - \beta$ is in $\mathbb{Q}$ then $f(x)$ is reducible." You can't assume multiple of them are in $\mathbb{Q}$.

14.6.13(b): 12 points.

(i) 4 points.

(ii) 4 points. Showing that $\mathbb{Q}(\alpha\beta) = \mathbb{Q}(\alpha^2)$ implies $b(a^2 - 4b)$ is a square is tricky. One option: first prove $\mathbb{Q}(\alpha\beta) = \mathbb{Q}(\alpha^2)$ implies $G \cong C$ directly, and then show that every Galois automorphism fixes $\sqrt{b(a^2 - 4b)}$.

(iii) 4 points.