Homework 18, due Monday March 28

MAT347

1. For any positive integer n, let L_n be the splitting field of $X^n - 2$ over $\mathbb{Q}$.
   (a) Prove that $L_n = \mathbb{Q}(\sqrt[n]{2}, \zeta_n)$, where $\zeta_n = e^{2\pi i/n}$. Prove that $|L_n : \mathbb{Q}| \leq n\varphi(n)$, where φ is the Euler function.
   (b) Find one value of n for which $|L_n : \mathbb{Q}| \neq n\varphi(n)$.
   (c) Let p be a prime. Prove that $|L_p : \mathbb{Q}| = p(p-1)$.
   (d) Prove that $\mathrm{Gal}(L_p/\mathbb{Q})$ is isomorphic to the holomorph of a cyclic group of order p. (For the definition of the holomorph of a group, see Example 5 on page 179 of the book.)

2. Recall that a field extension K/F is *2-filtered* when there are subextensions $F = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m = K$ such that $|K_i : K_{i-1}| = 2$ for all i.

   Let $z \in \mathbb{C}$. Consider the following three statements:

   (i) The point z is constructible with straightedge and compass starting from the points 0 and 1 in $\mathbb{C}$.
   (ii) There is a 2-filtered field extension $K/\mathbb{Q}$ such that $z \in K$.
   (iii) The field extension $\mathbb{Q}(z)/\mathbb{Q}$ is 2-filtered.

   We have proven in class that $(i) \iff (ii)$. Clearly $(iii) \implies (ii)$. The goal of this problem is to prove that $(ii) \implies (iii)$.

   (a) Let $\mathbb{Q} \subseteq M \subseteq L$ be field extensions and assume that $L/\mathbb{Q}$ is 2-filtered and normal. Prove that $M/\mathbb{Q}$ is 2-filtered.
   Hint: Use the Fundamental Theorem of Galois Theory to translate this question into a problem about group theory. Then realize that you need to prove a lemma about groups. Do so. (Note: once you have translated the question into a problem about groups, you still have quite a bit of work to do. It will help to remember what you know about the centre of a 2-group.)
   (b) Prove that the normal closure of a 2-filtered field extension is 2-filtered.
   Hint: From muy notes, use Propositions 3.8 (every field extension of order 2 is obtained by "adding a square root") and 5.16 (the normal closure of a field extension is the composite of finitely many isomorphic copies of it).

(c) Prove that $(ii) \implies (iii)$ in the above statements.

3. Derive completely an expression for $\cos \frac{2\pi}{17}$ in terms of rational numbers, square roots, and the field operations. (This expression appears on page 602 of the book, and the process is outlined.)