MAT 1060H1F
Assignment 7

Prof. McCann Due: noon Thursday Nov. 4

We will complete Evans 3.4 this week and proceed to Evans Chapter 5 next week.

To be handed in: Evans (Second edition) # 3.16, 3.17, 3.19, 3.20 plus

5. VISCOUS FINGERING: Imagine a layer of heavy fluid of density ρ_0 floating on top of a light fluid of density $\rho_1 < \rho_0$. Any slight perturbation of the flat interface between the two immiscible fluids will destabilize the situation, causing fingers of heavy fluid to penetrate into and displace the light fluid until eventually all of the heavy fluid has settled to the bottom, and all of the light fluid has risen to the top. The size of these fingers is determined by surface tension; in its absence the rate of growth of a finger is inversely proportionate to its thickness, leading to an ill-posed problem featuring complicated labyrithine geometry made up of zillions of tiny fingers and filaments. Suppose we are interested not in the details of this geometry, but only in the time T which it takes for the fluids to exchange places, and in the density profile $\rho(z,t) = (1-u(z,t))\rho_0 + u(z,t)\rho_1$ at each height $z \in [-1,1]$ in the *mixing layer* at time $0 < t < T$, and the overall thickness of this layer $\{z \in [-1,1] \mid 0 < u(z,t) < 1\}$.

In a seminal work (Comm. Pure Appl. Math. **52** (1999) 873–915), F. Otto proposed that the evolution of $u(z,t)$ would be governed by the conservation law

$$0 = \frac{\partial u}{\partial t} + \frac{\partial}{\partial z}[u(1-u)]$$

subject to the initial conditions

$$u(z,0) = \begin{cases} 0 \text{ if } z \in [z_0, +1] \\ 1 \text{ if } z \in [-1, z_0] \end{cases}$$

and boundary conditions

$$u(t,z) = \begin{cases} 1 \text{ if } z > +1 \\ 0 \text{ if } z < -1. \end{cases}$$

Solve the conservation law to find the mixing time T, the thickness of the mixing layer, and the density profile at each time $0 < t < T$ in the symmetric case $z_0 = 0$.