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# A tale of three eras: The discovery and rediscovery of the Hungarian Method ☆

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ABSTRACT

In the Fall of 1953, a translation of a paper of Jenő Egerváry from Hungarian into English combined with a result of Dénes Kőnig provided the basis of a good algorithm for the Linear Assignment Problem. To honor the Hungarian mathematicians whose ideas had been used, it was called the Hungarian Method. In 2005, Francois Ollivier discovered that the posthumous papers of Jacobi contain an algorithm that, when examined carefully, is essentially identical to the Hungarian Method. Since Jacobi died in 1851, this work was done over a hundred years prior to the publication of the Hungarian Method in 1955. This paper will provide an account of the mathematical, academic, social and political worlds of Jacobi, Kőnig, Egerváry, and Kuhn. As sharply different as they were (Prussian monarchy, Hungary under the Nazis and the Communists, and the post-war USA), they produced the same mathematical result. The paper is self-contained, assuming little beyond the duality theory of linear programming. The Hungarian Method and Jacobi's algorithm will be explained at an elementary level and will be illustrated by an example, solved both by the Hungarian Method and by Jacobi's Method.



## 1. Introduction

The subject of this paper is the tale of four mathematicians who worked in three very different periods. They are: Jacobi, who was active during the reigns of the Prussian monarchs, in the first half of the 19th century, Dénes Kőnig and Jenő Egerváry, who did their research in Hungary in the first half of the 20th century, when it was governed by Admiral Horthy, the Germans and the Russians, and Harold Kuhn, who started his career in the post-war United States, during the period of Eisenhower, Nixon and McCarthyism.

These four mathematicians are connected by an algorithm, called the Hungarian Method which solves the mathematical problem that is known in Operations Research as the (Linear) Assignment Problem. We shall describe the role of each in the history of the Assignment Problem, while explaining the underlying mathematics.

## 2. Mid 20th century history

Our tale begins in 1953 and will move forward and backward in time. I spent the summer of 1953 at the Institute for Numerical Analysis [1] at the University of California at Los Angeles. Although I was supported by the National Bureau of Standards, I had no fixed duties and spent most of the summer working on the Traveling Salesman Problem and the Assignment Problem.

It is important to establish the historical context of this work. Linear programming was about 6 years old [2]. In the summer of 1948, Gale, Kuhn and Tucker had formed a project to study the relationship between linear programming and two person matrix games. An important side product was the first rigorous proof of the Duality of Linear programming [3]. In the summer of 1950, Kuhn and Tucker presented the first paper on Nonlinear Programming [4]. Although the Simplex Method was being programmed for the new electronic computing machines, the possibility of special algorithms for problems like the Assignment Problem seemed to be a real opportunity.

Happily, there is a superb new book [5] that gives a detailed history of the sources of the Assignment Problem and its subsequent development. This will allow us to give a very compressed treatment of the problem which can be stated as follows: the data are the real number entries $r_{i,j}$ of a square matrix R. Find $n$ entries, one in each row and column, such that their sum is a maximum over all such "assignments". A natural interpretation is that $r_{i,j}$ is the rating of individual $i$ in job $j$. Assign the individuals to the jobs so that the sum of the ratings is a maximum.

☆ This is an edited version of the plenary address delivered on July 13, 2010 at EURO XXIV in Lisbon, Portugal. Explanatory footnotes and references are signaled in the text by "[n]". A video of the lecture, which contains additional graphical material, is available on the Web at http://www.euro2010lisbon.org/?content=movie.

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### 2.1. Why is the Assignment Problem a linear program?

The assignments can be represented as square matrices with entries 0 or 1, and with one 1 in each row and column. Such matrices are called permutation matrices and, geometrically, are points in a space with $n^2$ dimensions. Since our objective is to maximize a linear function (the "rating sum") over these points, we get the same answer if we maximize over the convex hull of these points. Here we are aided by a theorem [6] that is implicit in the work of Kőnig, but which is usually credited to a 1946 paper of Garrett Birkhoff.

The convex hull of the permutation matrices is the set of doubly stochastic matrices (nonnegative matrices with row and column sums all equal to one).

Thus we can state the linear program that is the same as the Assignment Problem and its dual program [7].

Maximize the sum of $r_{i,j}$ times $x_{i,j}$ over $i$ and $j$ subject to the constraints that the row and column sums of the matrix $x_{i,j}$ are all 1 and that the entries $x_{i,j}$ are nonnegative.

The dual program is:

Choose dual variables $u_i$ and $v_j$, such that $u_i + v_j$ is greater or equal to $r_{i,j}$ for all $i$ and $j$ and so as to minimize the sum of the $u_i$ plus the sum of the $v_j$.

## 3. Konig's contribution

I knew all of these things at UCLA in 1953. Imagine my surprise when I encountered the following theorem in the classical book on graph theory published by Dénes Kőnig in 1936 [8]; I was reading a version (in German) published under the Alien Custodian Act in 1950. An English translation was published forty years later in 1990.

For a graph G let m be the (finite) minimal number of vertices with the property that every edge of G ends in one of these vertices, and let $n$ be the maximal number of edges which pair-wise have no common endpoint. If G is a bipartite graph then $m = n$. Recall that a bipartite graph is a graph in which the vertices fall into two sets such that every edge of the graph joins a vertex in one set to a vertex $i$ the other.

A bipartite graph can be represented by a matrix in which the two sets of vertices are the rows and columns and an edge joining row $i$ to column $j$ is shown by a 1 in position $(i,j)$ and zeros elsewhere. Then we have:

Given a square matrix with entries zero or one, the maximum number of ones that can be chosen, no two in the same line (row or column) is equal to the minimum number of lines needed to cover all of the ones.

This is an astonishing result, given the time it was discovered by Kőnig. It is first of all an example of linear programming duality proved some decades before Dantzig had formulated linear programming, It is also the first example of a problem in combinatorial optimization that was solved by a constructive polynomial time algorithm laid out by Kőnig. You can understand my excitement as I returned to Bryn Mawr at the end of the summer, knowing that all one had to do was to reduce the general problem to a series of 0–1 problems. But, before I tell you the rest of the story, I must tell you [9].

## 4. Who was Dénes konig?

First, I shall teach you how to spell Kőnig's name correctly, something that is seldom done in the literature. The diacritical mark above the "o" in Kőnig is a Hungarian "double acute" not the "umlaut" that appears in the German word for "king". There is a letter from Kőnig [10] that asserts that this is the correct spelling.

Kőnig was born in Budapest on September 21, 1884 to a family of Jewish origin but was baptized as a Christian. His father, Gyula (or Julius) Kőnig, was an important mathematician, whose influence on Hungarian mathematics extended into the 20th century. Dénes was a boy prodigy, winning highly competitive prizes and publishing his first book on Mathematical Recreations while still in a Budapest Gymnasium.

His further mathematical training was done half in Budapest and half at Göttingen, where he was introduced to topological concepts by Minkowski. Having finished his studies in 1907, he worked as Demonstrator at the Polytechnic University of Budapest (BME). He remained there until his death in 1944, attaining the rank of Professor in 1935.

Of course, Kőnig's name is known to most of you through his work on graph theory. His book, which I was reading in 1953, was the only book length treatment of graph theory until the publication of Claude Berge's book [11] in 1958. Through his lectures Kőnig inspired an entire generation of Hungarian mathematicians to work in graph theory, including Egerváry, Erdős, Turán, Gallai, and many others. One can see a direct line of influence, from Kőnig through Gallai to Lovász and the current world class Hungarian school of combinatorics.

Several sources tell of Kőnig's efforts to help persecuted Jewish mathematicians. I can document one moving example. In November 1941, he wrote a letter [12] to John von Neumann on behalf of the 15 years old Peter Lax, who was leaving Hungary for America with his family. In the letter, he writes of "the boy's extraordinary mathematical ability" and says that it is "in the common interest to further cultivate and support this extraordinary talent". In conclusion, he urges von Neumann "to take him in your benevolent protection and appreciate him as a future scientist". In writing this letter, Kőnig not only demonstrated his generous humanity but also his ability in recognizing talent. Von Neumann took the young Peter under his wing and Lax went on to become a distinguished mathematician, winning both the Wolf and Abel prizes.

The letter from Kőnig to von Neumann contains one other vital piece of information, namely his address: "Horthy Miklós út 28". Kőnig lived on this busy cobble-stoned street that leads from the Hotel Gellert on the Danube into Buda. It was named for Miklós Horthy, the Regent of Hungary from 1920 to 1944. Horthy Miklós út was renamed Bartók Béla út after the Second World War so that the apartment house in which Kőnig lived in 1941 is now Bartók Béla út 28. It is a comfortable apartment house, no more than a five minute walk to the BME where Kőnig taught. When Bartók died in New York in 1945, he left a condition in his will that no place in Budapest should be named after him, so long as there were still places named Hitler and Mussolini. These have been renamed so his condition was satisfied.

Hunting for details about the building in which Kőnig lived, I located a genealogical website [13] that tells the story of an Alsatian couple who visited Budapest three times in the 30s. The couple stayed several times in a boarding house: the Hadik Pensio at Horthy Miklós út 28, the last time in 1938. In the account. written by their son, it is asserted that "The vast building contained the Hadik Coffee House, a famous prewar literary café which was frequented by, among others, Arthur Koestler". The accompanying photo suggests that the "vast building" was not 28 but 36, which has been restored in the last few years, with the Hadik Coffee House again on the ground floor.

Tibor Gallai, whose thesis was supervised by Kőnig and who is a principal biographical source, has written [14]: ‘‘He liked his colleagues and was an indispensible participant in the coffee house meetings of mathematicians’’. One can easily imagine that the coffee house in question was the Hadik and Kőnig, as a bachelor, took meals at the associated cafe.

Again, Gallai has written: ‘‘Kőnig was a cheerful, sociable man with sparkling humor, who enjoyed telling anecdotes’’. Here is one involving Kőnig, that was told to me by Peter Lax [15]: The German version of the famous dictum ‘‘In mathematics there is no royal road’’ is: ‘‘In der Mathematik gibt kein Königsweg’’. Someone quoted this to Dénes by way of teasing him, to which he replied,: ‘‘Aber es gibt ein Grafenweg’’. (The word ‘‘Graf’’ also means the aristocratic title ”count”.)

The events leading up to Kőnig’s death are closely intertwined with the history of Hungary at the end of the Second World War. German forces occupied Hungary in March, 1944, after attempts by Hungary to negotiate an armistice with the western Allies. By the end of July, nearly 440,000 Jews from the countryside had been deported from Hungary. The only substantial Jewish community left in Hungary was that of Budapest. In August, 1944, Horthy attempted to reach an armistice with the Soviet Union whose army was on Hungary’s borders, He had begun final negotiations by mid-October, when the Germans sponsored a coup d’etat. They arrested Horthy and installed a new Hungarian government, lead by the fascist and radically anti-semitic Arrow Cross party on October 15, 1944.

Dénes Kőnig’s parents were of Jewish origin, but he was born a Christian and therefore exempted from wearing the Yellow Star of David before October 15. The apartment house at Horthy Miklós út 28 had been declared a ‘‘yellow star’’ Jewish house earlier on June 16 [16], a kind of mini-ghetto. According to a story told [17] by Pál Erdős, Kőnig committed suicide when ordered by the janitor to move to the ghetto. He jumped from a window to his death [18] on October 19, 1944. He was 60 years of age. On the next day, the systematic drive against the Jews of Budapest by the Arrow Cross began and all of the worst fears of Kőnig were realized.

## 5. Mid 20th century history continued

We now return to the more peaceful year 1953; as I read further in Kőnig’s book, I found a tantalizing footnote.

E. Egerváry gave a generalization of these theorems, Matrixok kombinstorius tulajdonsagáiról (Concerning combinatorial properties of matrices), Matemarikai és Fizikai Lapok, 38, 1931, pp. 16–28 (Hungarian with a German abstract).

These were the days before the ability of computers to do instant translations so, when I returned to Bryn Mawr in the fall, I went down the road to Haverford College and checked out a Hungarian grammar and a large Hungarian dictionary. I spent two weeks teaching myself Hungarian and then translated Egerváry’s paper [19]. Although he was not interested in constructing an algorithm, I found, to my delight, that the paper contained an idea that allowed me to reduce the general Assignment Problem to a finite sequence of 0–1 problems. The algorithm [20] that resulted can be described in a single page:

Hungarian Method

GIVEN R= ($r_{ij}$), CHOOSE DUAL VARIABLES SUCH THAT

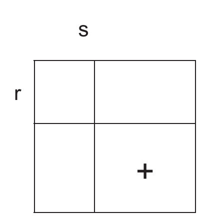
$w_{ij} = u_i + v_j - r_{ij} \geq 0$ FOR ALL $i$ AND $j$

AND WITH AT LEAST ONE 0 IN EVERY ROW AND COLUMN.

KŐNIG STEP

CONSTRUCT A MINIMAL COVER OF THE ZEROS IN W= ($w_{ij}$)
BY $r$ ROWS AND $s$ COLUMNS AND A SET OF $r$+$s$ ZEROS,
NO TWO IN THE SAME LINE (ROW OR COLUMN). IF $r+s = n$,
THEN WE ARE DONE. OTHERWISE, DO EGERVARY STEP:

EGERVÁRY STEP




MATRIX SHOWN IS W= ($w_{i,j}$)
WITH $r$ ROWS AND $s$ COLUMNS
IN THE MINIMAL COVER OF ZEROS.
LET $e$ BE THE MINIMUM OF THE
ENTRIES IN THE FIELD MARKED +.

CHANGE $v_j$ TO $v_j + e$ FOR $j$ IN THE COVER AND
CHANGE $u_j$ to $u_i - e$ FOR $i$ NOT IN THE COVER
$\sum u_i + \sum v_j$ DECREASES BY $(n - (r + s))\, e > 0$

REPEAT KŐNIG STEP!!!

The preparation of the problem is easily done. It consists in constructing a special feasible solution to the dual program. We first choose $v_j$ equal to the maximum entry in column $j$, for all $j$. We then choose $-u_i$ equal to the minimum entry in the row of entries $v_j - r_{i,j}$. This constructs a feasible solution ($w_{i,j}$ nonnegative) with a zero in every row and column of the matrix **W**.

We now perform a Kőnig step, choosing a minimal cover of the zeros in **W** by $r$ rows and $s$ columns and a set of $r + s$ zeros, no two in the same row or column. If $r + s$ is equal to $n$, then we have an optimal assignment. Otherwise we perform an Egerváry step, changing the dual variables as shown. The algorithm then tells us to repeat a Kőnig step.

If the data are integers, this decreases the objective in the dual program by a positive integer amount. Since the objective function of the dual is an upper bound for all assignment sums. This guarantees the termination of the algorithm.

Before solving an example, I must tell you [21].

## 6. Who was Jenó Egerváry?

Jenő Egerváry was born in Debrecen on April 16, 1891 and thus was six and a half years younger than Kőnig. With the exception of a student study trip to Great Britain, all of his education and academic career took place in Budapest. He completed his degree under the supervision of Lipót Fejér in 1914 and began a teaching career at institutions with a distinctly applied flavor such as the Seismological Institute and the State Industrial College. After serving as privat-docent at several institutions, he was made a full professor at BME in 1941.

After the Technical University split into two parts in 1955, Egerváry became head of the Department of Mathematics at the "Construction Industry and Transportation Engineering" Technical University until he was forced to retire on October 15, 1958. The bulk of Egerváry's research has an applied flavor, including several papers dealing with mathematical models for suspension and chain bridges, which may have been applied to the rebuilding of the Széchenyi Chain Bridge, one of Budapest's most famous monuments.

Egerváry's paper of 1931, which I published in English in 1953, is Number 11 in a list of Egerváry's published papers. It was written shortly after Kőnig lectured on his duality result for the matching problem for bipartite graphs and exhibits Egerváry's virtuosity in matrix theory. He was forty years old when he wrote this paper. I am very pleased by the fact that Egerváry learned about the Hungarian Method in 1957 and that it stimulated research related to the Hungarian Method in the last year of his life.

The background of Egerváry's suicide is this: In order to improve the financial situation of the faculty and staff of the University, the Government allowed some of them to enter into contracts with industrial enterprises to solve scientific and technical problems. A certain portion of the money that was paid for it was distributed among the participants. This method of obtaining extra income was new for the participants and the system was not supervised carefully. It is possible that some payments were made before the work was completed. It may be that some of the faculty, who had no previous experience in such matters, were guilty of irregularities in the book-keeping. Therefore the Ministry of Education prescribed a general revision of the bookkeeping in 1958. They inflicted an exemplary punishment on several Departments. Egerváry's department was among them. His deputy professor, Victor Lovass-Nagy, was arrested. I have learned from Lovass-Nagy's widow [22] that he was in prison on the day that their daughter was born, September 17, 1958. His sentence was four years and he served two years.

The pressure on Egerváry was intense. Several departments were dismantled and he was forced to retire. His assistant was in prison. There was the prospect of a demeaning trial and the end of a distinguished academic career. Nevertheless, he continued his academic activity to the end of his life. On 28 November, 1958, Egerváry gave a lecture at the Hungarian Academy of Sciences. It was a very good lecture delivered in his usual elegant way. However there was something remarkable in his mood and how he was dressed (he wore a black suit and black necktie).

Two days later, Egerváry committed suicide. The last evening before his death he asked his wife for a strong sleeping pill and asked her not to disturb him. When she went to sleep, he opened his veins. Next morning when she found him, he had lost so much blood that he could not be revived, The police came, searched the flat and confiscated many documents including a farewell letter. Egerváry's wife had read the letter and reported that it said that he was afraid that he would be the subject of a show trial and for this reason committed suicide.

Egerváry's suicide elicited shock and disbelief among his colleagues and students that lasts to this day. Given his outstanding record and position (elected as a full member of the Hungarian Academy of Sciences in 1946, winner of the Kossuth Prize on two occasions, co-founder of the Institute of Applied Mathematics in the Academy), they believe that he would have escaped any serious punishment.

After years of neglect and near invisibility, there have been notable efforts in the last few years to honor Egerváry and restore the reputation he deserves. The Hungarian Operations Research Society (HOROS) organized a memorial session for Egerváry at their 25th annual conference and established the Egerváry medal for lifetime OR achievement in 2004. A statue of Egerváry was erected on the campus of BME in 2006. Two parallel events took place on the 50th anniversary of his death in 2008, the first sponsored by the Italian Operational Research Society in Ischia, the second was a special memorial workshop organized by the HOROS and the Hungarian Academy of Sciences. The Special Issue of the Central European Journal of Operations Research published this year [23] contains a selection of lectures presented at these parallel events.

I cannot leave this brief sketch of Egerváry's life and times without mentioning that he was an avid mountain climber. On August 19th, 1920, he scaled Mount Gerlach, the highest mountain in the Carpathian range.

## 7. An example

It is time to see the Hungarian Method in action. Here we have the rating matrix for a 4 by 4 example

| | | | |
|---|---|---|---|
| 8 | 7 | 9 | 9 |
| 5 | 2 | 7 | 8 |
| 5 | 1 | 4 | 8 |
| 2 | 2 | 2 | 6 |

We set $v_j$ to be the maximum in column $j$ for each $j$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0 | 0 | 0 | 0 | |
| | 5 | 2 | 7 | 8 | W= | 3 | 5 | 2 | 1 | |
| | 5 | 1 | 4 | 8 | | 3 | 6 | 5 | 1 | |
| | 2 | 2 | 2 | 6 | | 6 | 5 | 7 | 3 | |

This creates a zero in every column of **W**.

We now set $u_l$ to be the negative of the minimum entry in every row in which **W** has no zero.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0 | 0 | 0 | 0 | |
| | 5 | 2 | 7 | 8 | W= | 3 | 5 | 2 | 1 | |
| | 5 | 1 | 4 | 8 | | 3 | 6 | 5 | 1 | |
| | 2 | 2 | 2 | 6 | | 6 | 5 | 7 | 3 | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 0 | ◄ |
| -1 | 5 | 2 | 7 | 8 | W= | 2 | 4 | 1 | 0* | |
| -1 | 5 | 1 | 4 | 8 | | 2 | 5 | 4 | 0 | |
| -3 | 2 | 2 | 2 | 6 | | 3 | 2 | 4 | 0 | |
| | | | | | | | | | ▲ | |

W now has a zero in every row and column. We now perform a Kőnig step, choosing as the cover of the first row and last column and the partial assignment of the two zeros marked with asterisks.

Now an Egerváry step ($v_4$ goes up by 1 and $u_2$, $u_3$, and $u_4$ go down by 1) followed by a Kőnig step.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0 | 0 | 0 | 0 | |
| | 5 | 2 | 7 | 8 | | 3 | 5 | 2 | 1 | |
| | 5 | 1 | 4 | 8 | | 3 | 6 | 5 | 1 | |
| | 2 | 2 | 2 | 6 | | 6 | 5 | 7 | 3 | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 0 | ◄ |
| -1 | 5 | 2 | 7 | 8 | W= | 2 | 4 | 1 | 0* | |
| -1 | 5 | 1 | 4 | 8 | | 2 | 5 | 4 | 0 | |
| -3 | 2 | 2 | 2 | 6 | | 3 | 2 | 4 | 0 | |
| | | | | | | | | | ▲ | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 10 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 1 | ◄ |
| -2 | 5 | 2 | 7 | 8 | W= | 1 | 3 | 0* | 0 | ◄ |
| -2 | 5 | 1 | 4 | 8 | | 1 | 4 | 3 | 0* | |
| -4 | 2 | 2 | 2 | 6 | | 2 | 1 | 3 | 0 | |
| | | | | | | | | | ▲ | |

Another Egerváry step followed by a Kőnig step.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | |
| | 8 | 7 | 9 | 9 | | 0 | 0 | 0 | 0 | |
| | 5 | 2 | 7 | 8 | W= | 3 | 5 | 2 | 1 | |
| | 5 | 1 | 4 | 8 | | 3 | 6 | 5 | 1 | |
| | 2 | 2 | 2 | 6 | | 6 | 5 | 7 | 3 | |

| | 8 | 7 | 9 | 9 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 0 | ◀ |
| -1 | 5 | 2 | 7 | 8 | W= | 2 | 4 | 1 | 0* | |
| -1 | 5 | 1 | 4 | 8 | | 2 | 5 | 4 | 0 | |
| -3 | 2 | 2 | 2 | 6 | | 3 | 2 | 4 | 0 | |
| | | | | | | | | | ▲ | |

| | 8 | 7 | 9 | 10 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 1 | ◀ |
| -2 | 5 | 2 | 7 | 8 | W= | 1 | 3 | 0* | 0 | ◀ |
| -2 | 5 | 1 | 4 | 8 | | 1 | 4 | 3 | 0* | |
| -4 | 2 | 2 | 2 | 6 | | 2 | 1 | 3 | 0 | |
| | | | | | | | | | ▲ | |

| | 8 | 7 | 9 | 11 | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 2 |
| -2 | 5 | 2 | 7 | 8 | W= | 1 | 3 | 0* | 1 |
| -3 | 5 | 1 | 4 | 8 | | 0 | 3 | 2 | 0* |
| -5 | 2 | 2 | 2 | 6 | | 1 | 0* | 2 | 0 |

and we have the optimal assignment indicated by the 4 asterisks in W.

## 8. Mid 20th century history concluded

In the fall of 1953, I solved a number of 12 by 12 Assignment Problems with 3 digit integer entries by hand in under 2 hours each and was convinced that the method was better than the Simplex Method. In a later paper [24] on the history of the Hungarian Method, I expressed my conviction in the following terms: “This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world”. I also asserted that “the 10 by 10 Assignment Problem is a linear program with 100 nonnegative variables and 20 equation constraints (of which only 19 are needed). In 1953, there was no machine in the world that had been programmed to solve a linear program that large!”.

I have been taken to task by Schrijver [25] for this last statement when he wrote: “If ‘the world’ includes the Eastern Coast of the USA, there seems to be some discrepancy with remarks of Votaw” (in a 1952 paper to the effect that a 10 by 10 Assignment Problem had been solved with the Simplex Method on the SEAC in 20 minutes). I did not know that the SEAC had been programmed to execute the special version of the Simplex Method for the Transportation Problem (of which the Assignment Problem is a special and very degenerate case) and was only familiar with the limitations on the size of general linear programs that could be solved on the SEAC. Saul Gass has pointed out that the IBM 701, unveiled to the public on April 7, 1953, could solve linear programs with $m$ (rows) <= 99, and $n$ (columns) unlimited; whether any $19 \times 100$ linear programs were solved by the Fall of 1953 is not known.

In preparing for this lecture, Saul Gass kindly provided me with all of the available written material (published and unpublished) on the solution of linear programs on the SEAC. I find nothing in them that would contradict the first statement quoted above. Concerning the superiority of the Hungarian Method over the Simplex Method, there is the report [26] by Ford and Fulkerson in 1956: “The largest example tried was a $20 \times 20$ optimal Assignment Problem. For this example, the simplex method required well over an hour, the (Hungarian) method about thirty minutes of hand computation”.

Some people believed that the Hungarian Method was a variant of the Simplex Method; I won a 25 cent bet with Alan Hoffman when he admitted that this was not so.

The matter was laid to rest with the publication of the thesis [27] of Hans Joachim Schmid in 1974 that shows that the Hungarian Method is a long step greatest descent method in the space of the matrices W. Incidentally, this is a reference that is missing in the marvelous book [5] by Burkard, Dell’Amico, and Martello.

A living tribute to Egerváry is the existence of the Egerváry Research Group on Combinatorial Optimization (EGRES), led by András Frank and affiliated with Operations Research Department of Eötvös University (ELTE) and the Hungarian Academy of Sciences (MTA). This vibrant group carries on the ideas of Egerváry in the Budapest of today. To commemorate the $50^{th}$ anniversary of the publication of the Hungarian Method in 1955, EGRES organized a birthday conference [28] in Budapest. The gathering was held in the Hungarian Academy of Sciences on October 31, 2005.

## 9. Enter Jacobi

In early 2006, I received an astonishing email [29] from Francois Ollivier, a French mathematician whose area of research is differential equations. It said:

“Two years ago I began to study two old papers in Latin by Jacobi. These are related to a conjectural bound expressed by solving the Assignment Problem with a matrix $h_{ij}$ the order of variable $j$ in equation $i$. Jacobi gave a polynomial algorithm to compute the bound”. Then Ollivier asked me: “What is its relation to the Hungarian Method?” Before I give you a precise answer to Ollivier’s question, I would like to tell you [30].

## 10. Who was Jacobi?

Carl Gustav Jacob Jacobi was born in Potsdam on December 10th, 1804 to a wealthy and cultured Jewish family. The precocious boy was tutored by a maternal uncle and entered the Gymnasium in Potsdam at 12. Although he had been placed in the highest class in spite of his youth, he could not enter the University until he was 16. By then he excelled in Greek and Latin and had explored mathematics far beyond the school curriculum.

At the University of Berlin, Jacobi decided to concentrate on mathematics although the lectures in mathematics there were at a very elementary level at the time. He mastered the works of the leading mathematicians of the time on his own and, before he was 20, passed his preliminary examination for Oberlehrer. This gave him permission to teach not only mathematics but also Greek and Latin to high school students and ancient and modern history to junior high school students.

In spite of being a Jew, he was offered a position at a prestigious Gymnasium for the following summer. He had already submitted his Ph.D. thesis to the University and, upon application, began work on the Habilitation immediately. At this point, Jacobi became a Christian. Thus, he was able to begin a university career as Privatdozent at the University of Berlin at the age of twenty. Although his lectures were a success, there was no prospect of a salaried position in Berlin, and so Jacobi moved to the University of Königsberg in the spring of 1826.

Most of you know Jacobi through the concepts of Jacobian matrix or Jacobian determinant; however he was proficient in many fields of mathematics and published prolifically. At Königsberg, Jacobi lectured about 10 hours a week and, in a manner that was unusual for the time, discussed his recent research openly. He introduced the research seminar involving a group of advanced students.

In 1843, Jacobi was diagnosed with a severe case of diabetes. His doctor prescribed a trip to Italy in the hope that the mild climate would improve his health. Returning to Prussia, Jacobi lectured occasionally at the University of Berlin, supported by a allowance provided by Frederick William IV, who was the King of Prussia for over twenty years. This was essential because the fortune that Jacobi had inherited from his father had disappeared in the depression that swept Europe in the 1840's. He was supporting his wife, his children and his mother.

At this point Jacobi made a serious mistake, he entered politics. According to Bell, who is notoriously unreliable, he did so on the advice of his physician "to benefit his nervous system". Be that as it may, there is a record [31] of Jacobi attending a Masonic revolutionary conference in Strasburg in 1847. More publicly, he ran for Parliament and signed a petition in 1848 asking for the King to turn over his power to the Parliament. Naturally Frederick William was not happy with this. He threatened to cut off Jacobi's allowance but, when Jacobi received an offer from Austria, the King relented and Jacobi stayed on in Berlin.

One might have expected Jacobi to die of overwork; however his death was much more prosaic. Early in 1851, he contracted influenza. Barely recovered, he developed smallpox and died within a week on the 18th of February, 1851. He was 46 years old.

## 11. Jacobi's method

Jacobi was astonishingly modern, acting as if he knew that computers would be invented some day. The name given to him by Bell, "The great Algorist" is richly deserved. To evaluate Jacobi's achievement, it is important to observe that it was made without suitable terminology and notation. The term "matrix" was only introduced by Sylvester [32] in 1850, a year before Jacobi's death and surely after this work was done. In spite of these handicaps, it is clear that Jacobi started with two elementary and crucial observations:

(1) The location of an optimal assignment is not changed if we add a constant to all of the entries in a row of the ratings matrix.
(2) If column maxima can be chosen in every column, with no two in the same row, they constitute an optimal assignment.

With these two observations, here is Jacobi's algorithm on one page:

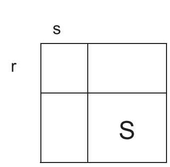
JACOBI'S ALGORITHM

GIVEN R= ($r_{ij}$), UNDERLINE THE MAXIMA IN EACH COLUMN. IN EACH ROW WITHOUT A COLUMN MAXIMUM, CHOOSE A NUMBER $u_i$ TO ADD TO ALL OF THE ENTRIES IN ROW SO AS TO CREATE AT LEAST ONE COLUMN MAXIMUM IN EVERY ROW AND COLUMN OF R. UNDERLINE THE NEW MAXIMA.

JACOBI/ KŐNIG STEP

CONSTRUCT A MINIMAL COVER OF THE COLUMN MAXIMA IN ($r_{ij} + u_i$) BY r ROWS AND s COLUMNS AND A SET OF $r + s$ COLUMN MAXIMA, NO TWO IN THE SAME LINE (ROW OR COLUMN). IF $r + s = n$ THEN WE ARE DONE. OTHERWISE, DO:

JACOBI/EGERVÁRY STEP




MATRIX SHOWN IS ($r_{ij} + u_i$) WITH $r$ ROWS AND $s$ COLUMNS IN THE COVER OF THE COLUMN MAXIMA. ADD THE MINIMUM NUMBER $t$ TO ALL OF THE $u_i$ IN ROWS i NOT IN THE COVER TO CREATE AT LEAST ONE NEW COLUMN MAXIMUM IN THE REGION S. UNDERLINETHE COLUMN MAXIMA.

REPEAT JACOBI/ KŐNIG STEP!!!

We start by underlining all of the column maxima in each of the columns. Then, in each row without a column maximum, we add a constant to all of the entries so as to create a column maximum in that row. Then we underline the new maxima.

We now perform a Kőnig step, with the underlined elements playing the role of the zeros in the Hungarian Method. If we can choose an underlined element in every column with no two in the same row, we are done. Otherwise, we perform Jacobi's version of the Egerváry step.

There are no column maxima in the region S of the matrix consisting of entries in rows and columns not in the cover. Find the minimum number $t$ to add to these entries to create a new column maximum and add it to all entries in rows not in the cover.

We now repeat a Jacobi/Kőnig step.

To show you that this is essentially the same as the Hungarian Method, let us solve the example again and compare the results.

$$\begin{array}{|cccc|}\hline \underline{8} & \underline{7} & \underline{9} & \underline{9} \\ 5 & 2 & 7 & 8 \\ 5 & 1 & 4 & 8 \\ 2 & 2 & 2 & 6 \\ \hline \end{array}$$

Here we have underlined each of the column maxima.

$$\begin{array}{|cccc|}\hline \underline{8} & \underline{7} & \underline{9} & \underline{9} \\ 5 & 2 & 7 & 8 \\ 5 & 1 & 4 & 8 \\ 2 & 2 & 2 & 6 \\ \hline \end{array}$$

$$\begin{array}{c|cccc|c} \cline{2-5} & \underline{8} & \underline{7} & \underline{9^*} & \underline{9} & \blacktriangleleft \\ 1 & 6 & 3 & 8 & \underline{9^*} & \\ 1 & 6 & 2 & 5 & \underline{9} & \\ 3 & 5 & 5 & 5 & \underline{9} & \\ \cline{2-5} \multicolumn{1}{c}{} & & & & \blacktriangle & \end{array}$$

Here we have added 1,1, and 3 to rows 2, 3 and 4 to create column maxima in these rows, then constructed a minimal cover of two lines indicated by the arrows and chosen two underlined elements, indicated by the asterisks.

$$\begin{array}{|cccc|}\hline \underline{8} & \underline{7} & \underline{9} & \underline{9} \\ 5 & 2 & 7 & 8 \\ 5 & 1 & 4 & 8 \\ 2 & 2 & 2 & 6 \\ \hline \end{array}$$

$$\begin{array}{c|cccc|c} \cline{2-5} & \underline{8} & \underline{7} & \underline{9^*} & 9 & \blacktriangleleft \\ 1 & 6 & 3 & 8 & 9^* & \\ 1 & 6 & 2 & 5 & 9 & \\ 3 & 5 & 5 & 5 & 9 & \\ \cline{2-5} \multicolumn{1}{c}{} & & & & \blacktriangle & \end{array}$$

$$\begin{array}{c|cccc|c} \cline{2-5} & \underline{8^*} & \underline{7} & \underline{9} & 9 & \blacktriangleleft \\ 2 & 7 & 4 & \underline{9^*} & \underline{10} & \blacktriangleleft \\ 2 & 7 & 3 & 6 & \underline{10^*} & \\ 4 & 6 & 6 & 6 & \underline{10} & \\ \cline{2-5} \multicolumn{1}{c}{} & & & & \blacktriangle & \end{array}$$

Here we have added 1 to the rows not in the previous cover, then performed a Kőnig step, choosing 3 underlined elements and a cover of three lines

$$\begin{array}{|cccc|}\hline 8 & 7 & 9 & 9 \\ 5 & 2 & 7 & 8 \\ 5 & 1 & 4 & 8 \\ 2 & 2 & 2 & 6 \\ \hline \end{array}$$

| | | | | | |
|---|---|---|---|---|---|
| | 8 | 7 | 9* | 9 | ◄ |
| 1 | 6 | 3 | 8 | 9* | |
| 1 | 6 | 2 | 5 | 9 | |
| 3 | 5 | 5 | 5 | 9 | |
| | | | | ▲ | |

| | | | | | |
|---|---|---|---|---|---|
| | 8* | 7· | 9 | 9 | ◄ |
| 2 | 7 | 4 | 9* | 10 | ◄ |
| 2 | 7 | 3 | 6 | 10* | |
| 4 | 6 | 6 | 6 | 10 | |
| | | | | ▲ | |

| | | | | |
|---|---|---|---|---|
| | 8* | 7 | 9 | 9 |
| 2 | 7 | 4 | 9* | 10 |
| 3 | 8 | 4 | 7 | 11* |
| 5 | 7 | 7* | 7 | 11 |

After adding 1 to rows 3 and 4, we are able to choose 4 column maxima, with no two in the same line and so have an optimal assignment. It is instructive to look at this example solved by the Hungarian Method and by Jacobi's method side by side.

**HUNGARIAN** **JACOBI**

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 0 | | | 8 | 7 | 9 | 9 | |
| | 5 | 2 | 7 | 8 | W= | 3 | 5 | 2 | 1 | | | 5 | 2 | 7 | 8 | |
| | 5 | 1 | 4 | 8 | | 3 | 6 | 5 | 1 | | | 5 | 1 | 4 | 8 | |
| | 2 | 2 | 2 | 6 | | 6 | 5 | 7 | 3 | | | 2 | 2 | 2 | 6 | |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 9 | | | | | | | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 0 | ◄ | | 8* | 7 | 9 | 9 | ◄ |
| -1 | 5 | 2 | 7 | 8 | W= | 2 | 4 | 1 | 0* | | 1 | 6 | 3 | 8 | 9* | |
| -1 | 5 | 1 | 4 | 8 | | 2 | 5 | 4 | 0 | | 1 | 6 | 2 | 5 | 9 | |
| -3 | 2 | 2 | 2 | 6 | | 3 | 2 | 4 | 0 | | 3 | 5 | 5 | 5 | 9 | |
| | | | | | | | | | ▲ | | | | | | ▲ | |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 10 | | | | | | | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 1 | ◄ | | 8* | 7· | 9 | 9 | ◄ |
| -2 | 5 | 2 | 7 | 8 | W= | 1 | 3 | 0* | 0 | ◄ | 2 | 7 | 4 | 9* | 10 | ◄ |
| -2 | 5 | 1 | 4 | 8 | | 1 | 4 | 3 | 0* | | 2 | 7 | 3 | 6 | 10* | |
| -4 | 2 | 2 | 2 | 6 | | 2 | 1 | 3 | 0 | | 4 | 6 | 6 | 6 | 10 | |
| | | | | | | | | | ▲ | | | | | | ▲ | |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 8 | 7 | 9 | 11 | | | | | | | | | | | | |
| | 8 | 7 | 9 | 9 | | 0* | 0 | 0 | 2 | | | 8* | 7 | 9 | 9 | |
| -2 | 5 | 2 | 7 | 8 | W= | 1 | 3 | 0* | 1 | | 2 | 7 | 4 | 9* | 10 | |
| -3 | 5 | 1 | 4 | 8 | | 0 | 3 | 2 | 0* | | 3 | 8 | 4 | 7 | 11* | |
| -5 | 2 | 2 | 2 | 6 | | 1 | 0* | 2 | 0 | | 5 | 7 | 7* | 7 | 11 | |

First note that the patterns of zeros are identical to the pattern of underlined column maxima. Hence, if we use the same implementation of the Kőnig steps, the covers and asterisked choices are the same. Finally, the dual variables u are the negatives of the amounts that Jacobi adds to the rows to create new column maxima.

Hence we can conclude that the algorithms are the same, just expressed in different terms!

The Hungarian Method was not the only modern computer algorithm that Jacobi anticipated. In 1846, he formulated an algorithm [33] for finding the eigenvalues of real symmetric matrices to show that these values are real. Over a hundred years later, upon the arrival of automatic computation, the Jacobi method was soon rediscovered [34] by Goldstine, Murray and von Neumann. Jacobi called it "Ein leichtes Verfahren", that is, "An Easy Method". His original article, in German, is a masterpiece of applied mathematics, and deserves to be read by students today.

## 12. Conclusion

I believe that it is illuminating to explore the motivations of each of the characters in our story.

Jacobi was attempting to establish a bound on the degree of a system of differential equations and discovered the mathematical problem that is the Assignment Problem in the course of this research. The fact that he discovered a good algorithm for calculating this bound is thoroughly modern and out of context in the 19th century. Kőnig was investigating the problem of maximum matching, that is, the problem of finding the largest number of edges in a graph with no two edges having a vertex in common. He, too, was ahead of his time, when he discovered the duality result that holds when the graph is bipartite. Egerváry, as an expert in matrix theory, generalized Kőnig's result from 0–1 matrices to arbitrary real matrices, giving a constructive proof. Kuhn, armed with the weapon of linear programming duality at the beginning of the computer revolution, realized that there was latent in Kőnig and Egerváry's work a good algorithm for the Assignment Problem that could be programmed for electronic computation.

What conclusions can we draw from this story of the discovery and rediscovery of the Assignment Problem? There are two conclusions to be drawn that are rather obvious. First, the stories that have been told illustrate very clearly that mathematicians are human beings and live complicated human lives in the context of the times and the places in which they do their research. A second conclusion is that the mathematics that they discover is independent of the social and political environment in which it is discovered. Prussian monarchy, fascist and communist Hungary, and post-war America were all very different places but pieces of the same algorithm were discovered in each of them. Good, basic mathematics is independent of the social and political factors of the time in which it is discovered. Good mathematics is lying out there waiting to be discovered, and discovered, and discovered again.

## Acknowledgements

I would like to acknowledge the large number of friends and colleagues who have helped me in preparing this lecture: László Babai, András Frank, Charles Gillispie, Saul Gass, Alan Hoffman, Mihály Hujter, Jean Daniel Krebs, Peter Lax, Robert Leonard, Józsefne Libor, Klara Lovass-Nagy, László Lovász, Silvano Martello, András Prékopa, Pál Rosza, Hans Schneider, Vera Sós, Peter Szabó. Of course, they bear no responsibility for any errors.

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[9] In preparing this biographical sketch, the primary source was: The Life and Scientific work of Dénes Kőnig (1884-1944) by Tibor Gallai. This article is a translation by Emeric Deutsch and Peter Orlik of Tibor Gallai's ''Kőnig Dénes (1884–1944) which appeared in Matematikai Lapok, vol. 15, 1964, pp. 277–293. The translation was published in Linear Algebra and Its Applications, vol. 21, 1978, pp. 189–205. The translation contains two footnotes not in the Hungarian original.
[10] This letter was provided to me by Péter Gábor Szabó, who has published the correspondence of Kőnig with Laszlő Kalmár in ''KALMÁRIUM'', Polygon, Szeged, 2005, p. 476.
[11] Berge C., Théorie des graphes et ses applications. Collection Universitaire de Mathématiques, II, Dunod, Paris 1958, p. viii+277. (English edition, Wiley 1961, New York; Methuen & Co, London, 1962.) My major contribution to graph theory has been to persuade Allison Doig, of the London School of Economics, to translate Berge's book and to negotiate the agreement between Methuen and Wiley to publish it.
[12] This letter was provided to me by Peter Lax and translated from the Hungarian by Andras Prékopa. The letter has the additional endorsement: '' I join my friend's, Dénes Kőnig's request with the warmest support for Peter Lax whom I know very well''. Adolf Szűcs. As a young man, Andras Prékopa rented a room from Szűcs' widow.
[13] The URL of the reference to Horthy Miklos út 28 on the website is: <http://notresaga.com/voyage3.html>.
[14] See [9].
[15] Private communication.
[16] Tim Cole, Holocaust City, The Making of a Jewish Ghetto, Routledge, New York and London, 2003, p. 125.
[17] This account of Kőnig's suicide was told to Laszlo Babai by Paul Erdos. It is published in the article ''Paul Erdos has just left town'', available on Babai's website. It should be noted that a central ghetto was not established in Budapest until some months after Kőnig's suicide although his apartment house was designated a ''yellow star'' house [16] sometime before he died.
[18] This unpublished account of Kőnig's suicide was provided to me by Antal Varga, a retired historian of? mathematics in Szeged. He learned it as a private communication from Rózsa Péter and also from László Kalmár, both deceased.
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[21] In preparing this biographical sketch, the primary source was: Rósza, P. (1984) Jenő Egerváry: a great personality of the Hungarian mathematical school, Period Polytech Electr Eng 28: 287-298. The was supplemented by a substantial email exchange with Professor Rósza, who was a close friend and colleague of Egerváry. In addition, Vera Sos, the widow of Paul Turan, has provided the moving eye-witness account of Egerváry's last lecture.
[22] In several telephone conversations with Klara Lovass-Nagy, I learned that her husband, Victor, returned voluntarily from a lecture tour in Switzerland to face the charges, convinced of his innocence. When he was released from prison, he held an industrial position in Hungary, then taught at the University of Khartoum, Sudan before moving to Clarkson University, Potsdam, New York in 1966 where he was Professor of Mathematics and Electrical Engineering until his retirement in 1993.
[23] Central European Journal of Operations Research, Special issue in honour of Jeno Egerváry edited by T. Csendes, A. Galántai, and L. Szeidl, Volume 18, Number 1/March 2010.
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[28] The homepage of the birthday conference can be found at: <http://www.cs.elte.hu/egres/www/mf_events.html>.
[29] Olivier maintains a marvelous website: <http://www.lix.polytechnique/∼olivier/JACOBI/JACOBIEng/htm> which contains all of the relevant documents translated into English.
[30] In preparing this biographical sketch, the primary sources were the Encyclopedia Brittanica, the Dictionary of Scientific Biography, and the chapter on Jacobi in E.T. Bell's "Men of Mathematics".
[31] In <http://tcallenco.blogspot.com/2009_11_01_archive.html> we find "Plans for the next great revolution were made at the Masonic Congress of 1846 at Strasbourg. High-degree Freemasons and members of other secret societies attended this Congress. ... Among the German Freemasons' representatives were ... Karl Jacobi (a Jew and formerly a professor of mathematics at Konigsberg)". The source appears to be: Nesta H. Webster, World Revolution: The Plot Against Civilization, (London: Constable, 1921) p. 157.
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