Differentiable Manifolds

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The main references for these lecture notes are the first volume of Greub-Halperin-Vanstone, and the book by Bott-Tu (second edition!).

Contents

1. Manifolds

1.1. Definition of manifolds. A n-dimensional manifold is a space that locally looks like $\mathbb{R}^n$. To give a precise meaning to this idea, our space first of all has to come equipped with some topology (so that the word "local" makes sense). Recall that a *topological space* is a set M, together with a collection of subsets of M, called *open subsets*, satisfying the following three axioms: (i) the empty set $\emptyset$ and the space M itself are both open, (ii) the intersection of any finite collection of open subsets is open, (iii) the union of *any* collection of open subsets is open. The collection of open subsets of M is also called the topology of M. A map $f: M_1 \to M_2$ between topological spaces is called *continuous* if the pre-image of any open subset in M_2 is open in M_1. A continuous map with a continuous inverse is called a *homeomorphism*.

One basic ingredient in the definition of a manifold is that our topological space comes equipped with a covering by open sets which are homeomorphic to open subsets of $\mathbb{R}^n$.

Definition 1.1. Let M be a topological space. An *n-dimensional chart* for M is a pair (U, ϕ) consisting of an open subset $U \subset \mathbb{R}^n$ and a continuous map $\phi: U \to \mathbb{R}^n$ such that ϕ is a homeomorphism onto its image $\phi(U)$. Two such charts (U_α, ϕ_α), (U_β, ϕ_β) are *C^∞-compatible* if the *transition map*

$$\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism (a smooth map with smooth inverse). A covering $\mathcal{A} = (U_\alpha)_{\alpha \in A}$ of M by pairwise C^∞-compatible charts is called a *C^∞-atlas*.

Some people define a C^∞-manifold to be a topological space with a C^∞ atlas. It is more common, however, to restrict the class of topological spaces.

Definition 1.2 (Manifolds). A *C^∞-manifold* is a Hausdorff topological space M, with countable basis, together with a C^∞-atlas.

Remarks 1.3.
(a) We recall that a topological space is called *Hausdorff* if any two points have disjoint open neighborhoods. A *basis* for a topological space M is a collection $\mathcal{B}$ of open subsets of M such that every open subset of M is a union of open subsets in the collection $\mathcal{B}$. For example, the collection of open balls $B_\epsilon(x)$ in $\mathbb{R}^n$ define a basis. But one already has a basis if one takes only all balls $B_\epsilon(x)$ with $x \in \mathbb{Q}^n$ and $\epsilon \in \mathbb{Q}_{>0}$; this then defines a countable basis. A topological space with countable basis is also called *second countable*.
(b) The Hausdorff axiom excludes somewhat pathological examples, such as following: Let $M = \mathbb{R} \cup \{p\}$, where p is a point, with the topology given by open sets in $\mathbb{R}$, together with sets of the form $(U\backslash\{0\}) \cup \{p\}$, for open sets $U \subset \mathbb{R}$ containing 0. An open covering of M is given by the two sets $U_+ = \mathbb{R}$ and $U_- = \mathbb{R}\backslash\{0\} \cup \{p\}$. The natural projection from M to $\mathbb{R}$, taking p to 0, descends to smooth maps $\phi_+ : U_+ \to \mathbb{R}$ and $\phi_- : U_- \to \mathbb{R}$. Then M with atlas

$(U_\pm, \phi_\pm)$ satisfies all the axioms of a 1-dimensional manifold except that it is not Hausdorff: Every neighborhood of 0 intersects every neighborhood of p.

(c) It is immediate from the definitions that any open subset of a manifold is a manifold and that the direct products of two manifolds is again a manifold.

The definition of a manifold can be generalized in many ways. For instance, a *manifold with boundary* is defined in exactly the same way as a manifold, except that the charts take values in a half space $\{x \in \mathbb{R}^n |\, x_1 \geq 0\}$. For this to make sense, one needs to define a notion of smooth maps between open subsets of half-spaces of $\mathbb{R}^n, \mathbb{R}^m$: Such a map is called smooth if it extends to a smooth map of open subsets of $\mathbb{R}^n, \mathbb{R}^m$. Even more generally, one defines *manifolds with corners* modeled on open subset of the positive orthant $\{x \in \mathbb{R}^n |\, x_j \geq 0\}$ in $\mathbb{R}^n$.

A *complex manifold* is a manifold where all charts take values in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and all transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are *holomorphic*.

1.2. Examples of manifolds.

(a) *Spheres.* Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere of radius 1. Let $N = (1, 0, \ldots, 0)$ be the north pole and $S = (-1, 0, \ldots, 0)$ the south pole. Let $U_1 = S^n \backslash \{S\}$ and $U_2 = S^n \backslash \{N\}$. Define maps $\phi_j : U_j \to \mathbb{R}^n$ by

$$\phi_1(x) = \frac{x - (x \cdot N)N}{1 - x \cdot N}, \quad \phi_2(x) = \frac{x - (x \cdot S)S}{1 - x \cdot S} = \frac{x - (x \cdot N)N}{1 + x \cdot N}.$$

Then $\phi_j : U_j \to \mathbb{R}^n$ define the structure of an oriented manifold on S^n. Both charts are onto $\mathbb{R}^n$, and $\phi_1(U_1 \cap U_2) = \phi_2(U_1 \cap U_2) = \mathbb{R}^n \backslash \{0\}$. The inverse map to ϕ_1 reads,

$$\phi_1^{-1}(y) = \frac{(||y||^2 - 1)N + 2y}{1 + ||y||^2}.$$

Thus,

$$\phi_2 \circ \phi_1^{-1}(y) = \frac{y}{||y||^2},$$

a global diffeomorphism from $\mathbb{R}^n \backslash \{0\}$ onto itself. The 2-sphere S^2 is in fact a complex manifold: Identify $\mathbb{R}^2 \cong \mathbb{C}$ in the usual way, so that ϕ_1, ϕ_2 take values in $\mathbb{C}$. Replace ϕ_2 by its complex conjugate, $\overline{\phi}_2(x) = \overline{\phi_2(x)}$. In complex coordinates, $\overline{\phi}_2 \circ \phi_1^{-1}(z) = z^{-1}$, which is a holomorphic function. A different complex structure is obtained by replacing ϕ_1 by it complex conjugate. For $n \neq 2$, the spheres S^n are not complex manifolds.

(b) *Projective spaces.* Let $\mathbb{R}P(n)$ be the quotient $S^n / \sim$ under the equivalence relation $x \sim -x$. Let $\pi : S^n \to \mathbb{R}P(n)$ be the quotient map. For any chart $\psi : V \to \mathbb{R}^n$ of S^n with the property $x \in V \Rightarrow -x \notin V$, let $U = \pi(V)$, and $\phi : U \to \mathbb{R}^n$ the unique map such that $\phi \circ \pi = \psi$. The collection of all such charts defines an atlas for $\mathbb{R}P(n)$; the compatibility of charts follows from that for S^n.

(c) *Grassmannians.* The set $\mathrm{Gr}_{\mathbb{R}}(k,n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *Grassmannian of k-planes in* $\mathbb{R}^n$. A C^∞-atlas may be constructed as follows. For any subset $I \subset \{1,\ldots,n\}$ of cardinality $\#I = k$, let $\mathbb{R}^I \subset \mathbb{R}^n$ be the subspace consisting of all $x \in \mathbb{R}^n$ with $x_i = 0$ for $i \notin I$. Thus each $\mathbb{R}^I \in \mathrm{Gr}_{\mathbb{R}}(k,n)$. Let $U_I \subset \mathrm{Gr}(k,n)$ be the set of all k-dimensional subspaces $E \subset \mathbb{R}^n$ with $E \cap (\mathbb{R}^I)^\perp = \{0\}$. There is a bijection $\phi_I : U_I \cong L(\mathbb{R}^I, (\mathbb{R}^I)^\perp) \cong \mathbb{R}^{k(n-k)}$ of U_I with the space of linear maps $A_I : \mathbb{R}^I \to (\mathbb{R}^I)^\perp$, where each such A corresponds to the subspace $E = \{x + A_I(x) | \, x \in \mathbb{R}^I\}$.

To check that the charts are compatible, let Π_I denote orthogonal projection $\mathbb{R}^n \to \mathbb{R}^I$. We have to show that for all intersections, $U_I \cap U_{\tilde{I}}$, the map taking A_I to $A_{\tilde{I}}$ is smooth. The map A_I is determined by the equations

$$A_I(x_I) = (1 - \Pi_I)x, \quad x_I = \Pi_I x$$

for $x \in E$, and $x = x_I + A_I x_I$. Thus

$$A_{\tilde{I}}(x_{\tilde{I}}) = (I - \Pi_{\tilde{I}})(A_I + 1)x_I, \quad x_{\tilde{I}} = \Pi_{\tilde{I}}(A_I + 1)x_I.$$

The map $\Pi_{\tilde{I}}(A_I + 1)$ restricts to an isomorphism $S(A_I) : \mathbb{R}^I \to \mathbb{R}^{\tilde{I}}$. The above equations show,

$$A_{\tilde{I}} = (I - \Pi_{\tilde{I}})(A_I + 1)S(A_I)^{-1}.$$

The dependence of S on the matrix entries of A_I is smooth, by Cramer's formula for the inverse matrix. It follows that the collection of all $\phi_I : U_I \to \mathbb{R}^{k(n-k)}$ defines on $\mathrm{Gr}_{\mathbb{R}}(k,n)$ the structure of a manifold of dimension $k(n-k)$. Later, we will give an alternative description of the manifold structure for the Grassmannian as a "homogeneous space".

The discussion above can be repeated by replacing $\mathbb{R}$ with $\mathbb{C}$ everywhere. The space $Gr_{\mathbb{C}}(k,n)$ is a complex manifold of *complex* dimension $k(n-k)$, i.e. real dimension $2k(n-k)$. In particular, $\mathrm{Gr}_{\mathbb{C}}(1,n) = \mathbb{C}P(n)$ is the complex projective space.

(d) *Flag manifolds.* A (complete) flag in $\mathbb{R}^n$ is a sequence of subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n$ where $\dim V_k = k$ for all k. Let $\mathrm{Fl}(n)$ be the set of all flags. It is a manifold of dimension $(n^2 - n)/2$, as one can see from the following rough argument. Any real flag in $\mathbb{R}^n$ determines, and as determined by a sequence of 1-dimensional subspaces $L_1, \cdots, L_n$, where each L_j is orthogonal to the sum of L_k with $k \neq j$. Indeed the flag is recovered from this by putting $E_1 = L_1$, $E_2 = L_1 \oplus L_2$ and so on. There is an $\mathbb{R}P(n-1)$ of choices for L_1. Given L_1 there is an $\mathbb{R}P(n-2)$ of choices for L_2 (since L_2 is orthogonal to L_1). Given $L_1, \ldots, L_j$ there is an $\mathbb{R}P(n-j-1)$ of possibilities for L_{n+1}. Hence we expect,

$$\dim \mathrm{Fl}_{\mathbb{R}}(n) = \sum_{j=1}^{n-1} (n-j) = n(n-1)/2.$$

It is possible to construct an atlas for $\mathrm{Fl}_{\mathbb{R}}(n)$ using this idea. Below we will give an alternative approach, by showing that the flag manifold is a homogeneous space (see below). Similarly, one can define a complex flag manifold $\mathrm{Fl}_{\mathbb{C}}(n)$, consisting of flags of subspaces in $\mathbb{C}^n$. Also, one can define spaces $\mathrm{Fl}_{\mathbb{R}}(k_1, \ldots, k_l, n)$ of *partial* flags, consisting of subspaces $\{0\} = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = \mathbb{R}^n$ of given dimensions $k_1, \ldots, k_l, n$. Note that $\mathrm{Fl}_{\mathbb{R}}(k, n) = \mathrm{Gr}(k, n)$.

(e) *Klein Bottle.* Let M be the manifold obtained as a quotient $[0,1] \times [0,1]/\sim$ under the equivalence relation $(x, 0) \sim (x, 1),\ (0, x) \sim (1, 1-x)$. Exercise: The quotient space has natural manifold structure. Hint: Write M as a quotient of $\mathbb{R}^2$ rather than $[0,1]^2$. Then use charts for $\mathbb{R}^2$ to define charts for M.

A manifold M is called *orientable* if it admits an atlas such that the Jacobians of all transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ have positive determinants. An manifold M with such an atlas is called an *oriented* manifold.

Exercise 1.4. Show that $\mathrm{Gr}_{\mathbb{R}}(1, n+1) = \mathbb{R}P(n)$, and $\mathrm{Gr}_{\mathbb{R}}(k, n) = \mathrm{Gr}_{\mathbb{R}}(n-k, n)$.

Exercise 1.5. Construct a manifold structure on the space $M = \mathrm{Gr}^{or}_{\mathbb{R}}(k, n)$ of *oriented* k-planes in $\mathbb{R}^n$.

Exercise 1.6. Show that $\mathbb{R}P(n)$ is orientable if and only if n is odd. Any idea for which k, n the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k, n)$ is orientable? (Answer: If and only if n is odd.) Show that the Klein Bottle is non-orientable. Show that any complex manifold (viewed as a real manifold) is oriented.

1.3. Smooth maps between manifolds.

Definition 1.7. A map $F : N \to M$ between manifolds is called smooth (or C^∞) if for all charts (U, ϕ) of N and (V, ψ) of M with $F(U) \subset V$, the composite map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth. The space of smooth maps from N to M is denoted $C^\infty(N, M)$. A smooth map $F : N \to M$ with smooth inverse $F^{-1} : M \to N$ is called a *diffeomorphism*.

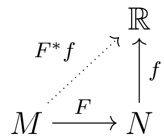
In the special case where the target space is the real line we write $C^\infty(M) := C^\infty(M, \mathbb{R})$. The space $C^\infty(M)$ is an algebra under pointwise multiplication, called the *algebra of functions on M*. For any $f \in C^\infty(M)$, one defines the *support* of f to be the closed set

$$\mathrm{supp}(f) = \overline{\{x \in M |\, f(x) \neq 0\}}.$$

Clearly, the composition of any two smooth maps is again smooth. In particular, any $F \in C^\infty(N, M)$ defines an algebra homomorphism

$$F^* : C^\infty(M) \to C^\infty(N), \quad f \mapsto F^* f = f \circ F$$

called the *pull-back*.



In fact, a given map $F : N \to M$ is smooth if and only if for all $f \in C^\infty(M)$, the pulled back map $F^*f = f \circ F$ is smooth. (Exercise.)

If M is a manifold, we say that a coordinate chart $\phi : U \to \mathbb{R}^m$ is centered at $x \in M$ if $\phi(x) = 0$.

DEFINITION 1.8. Let $F \in C^\infty(N, M)$ be a smooth map between manifolds of dimensions n, m, and $x \in N$. The *rank of F at x*, denoted $\mathrm{rank}_x(F)$, is the rank of the Jacobian

$$D_{\phi(x)}(\psi \circ f \circ \phi^{-1}) : \mathbb{R}^n \to \mathbb{R}^m,$$

for any choice of charts $\phi : U \to \mathbb{R}^n$ centered at x and $\psi : V \to \mathbb{R}^m$ centered at $F(x)$. The point x is called *regular* if $\mathrm{rank}_x(F) = m$, and *singular* (or *critical*) otherwise. A point $y \in M$ is called a *regular value* if $\mathrm{rank}_x(F) = m$ for all $x \in F^{-1}(y)$, *singular value* otherwise.

Note that the rank of the map F does not depend on the choice of coordinate charts. According to our definition, points that are not in the image of F are regular values.

LEMMA 1.9. *The map $M \to \mathbb{Z}$, $x \mapsto \mathrm{rank}_x(F)$ is lower semi-continuous: That is, for any $x_0 \in M$ there is an open neighborhood U around x_0 such that $\mathrm{rank}_x(F) \geq \mathrm{rank}_{x_0}(F)$ for $x \in U$. In particular, if $r = \max_{x\in M} \mathrm{rank}_x(F)$, the set $\{x \in M | \ \mathrm{rank}_x(F) = r\}$ is open in M.*

PROOF. Choose coordinate charts $\phi : U \to \mathbb{R}^n$ centered at x_0 and $\psi : V \to \mathbb{R}^m$ centered at $F(x_0)$. By assumption, the Jacobian $D_{\phi(x_0)} : \mathbb{R}^n \to \mathbb{R}^m$ has rank $r = \mathrm{rank}_{x_0}(F)$. Equivalently, some $r \times r$-minor of the matrix representing $D_{\phi(x_0)}$ has nonzero determinant. By continuity, the same $r \times r$-minor for any $D_{\phi(x)}$, $x \in U$ has nonzero determinant, provided U is sufficiently small. This means that the rank of F at x must be at least r. ∎

DEFINITION 1.10. Let $F \in C^\infty(N, M)$ be a smooth map between manifolds of dimensions n, m. The map F is called a

- *submersion* if $\mathrm{rank}_x(F) = m$ for all $x \in M$.
- *immersion* if $\mathrm{rank}_x(F) = n$ for all $x \in M$.
- *local diffeomorphism* if $\dim M = \dim N$ and F is a submersion (equivalently, an immersion).

Thus, submersions are the maximal rank maps if $m \leq n$, and immersions are the maximal rank maps if $m \geq n$.

THEOREM 1.11 (Local normal form for submersions). *Suppose $F \in C^\infty(N, M)$ is a submersion, $x_0 \in N$. Given any coordinate chart (V, ψ) centered at $F(x_0)$, one can find a coordinate chart (U, ϕ) centered at x_0 such that the map $\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by*

$$\tilde{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^m).$$

PROOF. The idea is simply to take the components of F as the first m components $\phi^j(x)$ of the coordinate function near x_0.

Using the given coordinate chart around $F(x_0)$ and any coordinate chart around x_0, we may assume that M is an open neighborhood of $0 \in \mathbb{R}^m$ and N an open neighborhood of $0 \in \mathbb{R}^n$. We have to find a smaller neighborhood U of $0 \in M \subset \mathbb{R}^n$ and a diffeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$ such that $\tilde{F} = F \circ \phi^{-1}$ has the desired form.

By a linear transformation of $\mathbb{R}^n$, we may assume that $\frac{\partial F^i}{\partial x^j}(0) = \delta_{ij}$ for $i, j \le m$. Let $\phi : M \to \mathbb{R}^n$ be the map

$$\phi(x^1, \ldots, x^n) = \big(F^1(x^1, \ldots, x^n), \ldots, F^m(x^1, \ldots, x^n), x^{m+1}, \ldots, x^n\big).$$

The Jacobian of ϕ at $x = 0$ is just the identity matrix. Hence the inverse function theorem applies: There exists some smaller neighborhood U of $0 \in M \subset \mathbb{R}^n$, such that ϕ is a diffeomorphism $U \to \phi(U)$. Then (U, ϕ) is the desired coordinate system. □

THEOREM 1.12 (Local normal form for immersions). *Suppose $F \in C^\infty(N, M)$ is an immersion, $x_0 \in N$. Given any coordinate chart (U, ϕ) centered at x_0, one can find a coordinate chart (V, ψ) centered at $F(x_0)$ such that the map $\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by*

$$\tilde{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

PROOF. The idea is to use the given coordinates on U as coordinates on $F(U)$, near $F(x_0)$, and supplementary $m - n$ coordinates in "transversal directions" to get coordinates on M near $F(x_0)$.

Using the given coordinates chart around x_0 and *any* coordinate chart around $F(x_0)$, we may assume that M is an open neighborhood of $0 \in \mathbb{R}^m$ and N an open neighborhood of $0 \in \mathbb{R}^n$. We have to find a smaller neighborhood $V \subset M$, and a diffeomorphism $\psi : V \to \psi(V) \subset \mathbb{R}^m$ such that $\tilde{F} = \psi \circ F$ has the form $\tilde{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$.

Let $x^1, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, \ldots, y^m$ the coordinates on $\mathbb{R}^m$. By assumption, the matrix $\frac{\partial F^i}{\partial x_j}(x)$ has maximal rank n for all $x \in U$. By a linear change of coordinates on V, we may assume that $(\frac{\partial F^i}{\partial x_j}(x_0))_{i,j \le n} = \delta_{ij}$. Consider the map Consider the map

$$\Psi : N \times \mathbb{R}^{m-n} \to \mathbb{R}^m, \ (x, s) \mapsto F(x) + (0, \ldots, 0, s^{n+1}, \ldots, s^m)$$

The Jacobian of Ψ at 0 is just the identity matrix. Hence, the inverse function theorem applies, and we can find an open neighborhood V around $0 \in M \subset \mathbb{R}^m$ such that Ψ^{-1} is a well-defined diffeomorphism over V. The map $\psi = \Psi^{-1} : V \to \psi(V) \subset U \times \mathbb{R}^{m-n} \subset \mathbb{R}^m$ is the desired coordinate chart. □

EXERCISE 1.13. Let $\pi : N \to M$ be a surjective submersion. Suppose $F : M \to X$ is any map such that F lifts to a smooth map $\hat{F} : N \to X$, i.e. $F \circ \pi = \hat{F}$. Then F is smooth.

DEFINITION 1.14. Let M be a manifold. A subset $S \subset M$ is called an *embedded submanifold* if for each $x_0 \in S$ there exists a coordinate system (U, ϕ) centered at x_0, such that

$$\phi(U \cap S) = \{(x^1, \ldots, x^m) \in \phi(U) |\ x^{k+1} = \ldots = x^m = 0\}$$

It is obvious from the definition that submanifolds inherit a manifold structure from the ambient space: Given a covering of S by coordinate charts (U, ϕ) as above, one simply takes $(U \cap S, \phi|_{U \cap S})$ to define an atlas for S.

The following two Theorems are immediate consequences of the normal form theorems for submersions and immersions, respectively.

THEOREM 1.15. *Let $F \in C^\infty(N, M)$ be a submersion. Then each level set $S = F^{-1}(y)$ for $y \in M$ is an embedded submanifold of dimension $n - m$.*

THEOREM 1.16. *Let $F \in C^\infty(N, M)$ be an immersion. For each $x_0 \in N$ there exists a neighborhood U of x_0 such that the image $S = F(U)$ is an embedded submanifold of dimension $m - n$.*

These theorems provide many new examples of manifolds. Often, manifolds are obtained as level sets for a smooth function on a Euclidean space $\mathbb{R}^N$. For example, we see again that $S^n \subset \mathbb{R}^{n+1}$ is a manifold. Another example is the 2-torus, for $0 < r < R$ the radii of the "small" and "big" circles, given as a level set $G^{-1}(r^2)$ where $G \in C^\infty(\mathbb{R}^3)$ is the function,

$$G(x^1, x^2, x^3) = (x^3)^2 + (\sqrt{(x^1)^2 + (x^2)^2} - R)^2.$$

The 2-torus can also be described as the image of an immersion $F : \mathbb{R}^2 \to \mathbb{R}^3$, where $F(\theta, \phi) = (x^1, x^2, x^3)$ is given by

$$\begin{aligned} x^1 &= (R + r\cos\theta)\cos\phi, \\ x^2 &= (R + r\cos\theta)\sin\phi, \\ x^3 &= r\sin\theta \end{aligned}$$

In fact, it is clear that this map descends to an embedding $\mathbb{R}^2/(2\pi\mathbb{Z})^2 \to \mathbb{R}^3$ as a submanifold.

We will show below that any compact manifold can be smoothly embedded into some $\mathbb{R}^N$. (In fact, compactness is not necessary but we won't prove this harder result.)

EXERCISE 1.17. Construct an explicit embedding of the Klein bottle into $\mathbb{R}^4$. Solution: Given $0 < r < R$ define $F(\theta, \phi) = (x^1, x^2, x^3, x^4)$ where

$$\begin{aligned} x^1 &= (R + r\cos\theta)\cos\phi, \\ x^2 &= (R + r\cos\theta)\sin\phi, \\ x^3 &= r\sin\theta\cos\phi/2, \\ x^4 &= r\sin\theta\sin\phi/2. \end{aligned}$$

for $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$.

1.4. Tangent vectors. There is a number of equivalent coordinate-free definitions for the tangent space T_xM of a manifold $x \in M$ at some point $x \in M$. Our favorite definition defines T_xM as the space of "directional derivatives".

DEFINITION 1.18. Let M be a manifold. A *tangent vector* at $x \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the "Leibnitz rule" (product rule)

$$v(f_1 f_2) = v(f_1)f_2(x) + f_1(x)v(f_2).$$

The vector space of tangent vectors at x is denoted T_xM, and called the *tangent space* at x.

It follows immediately from the definition that any tangent vector vanishes on constant functions. Indeed, if 1 denotes the constant function $f(x) = 1$, the product rule gives

$$v(1) = v(1 \cdot 1) = v(1) \cdot 1 + 1 \cdot v(1) = 2v(1)$$

thus $v(1) = 0$. Furthermore, the product rule shows that for any two functions g, h with $g(x) = h(x) = 0$, $v(gh) = 0$.

LEMMA 1.19. *If $U \subset \mathbb{R}^n$ is an open subset and $x_0 \in U$, the tangent space $T_{x_0}U$ is isomorphic to $\mathbb{R}^n$, with basis the derivatives in coordinate directions,*

$$\frac{\partial}{\partial x^i}|_{x_0} : f \mapsto \frac{\partial f}{\partial x^i}(x_0)$$

PROOF. We may assume $x_0 = 0$. Let $v \in T_0U$. Given $f \in C^\infty(U)$, use Taylor's theorem with remainder to write $f(x) = f(0) + \sum_{i=1}^m \frac{\partial f}{\partial x^i}(0)x^i + \sum_{i=1}^m g_i(x)x^i$, where g_i vanishes at 0. v vanishes on the constant $f(0)$, and also on the products $x_i g_i(x)$. Thus

$$v(f) = \sum_{i=1}^m \frac{\partial f}{\partial x^i}(0)v(x^i) = \sum_{i=1}^m a_i \frac{\partial f}{\partial x^i}|_{x=0}$$

where $a_i = v(x^i)$. □

LEMMA 1.20. *Let M be a manifold of dimension m. If $\iota : U \hookrightarrow M$ is any open neighborhood of x, the map*

$$\iota_* : T_xU \to T_xM, \ \iota_* v(f) = v(f|_U)$$

is an isomorphism. In particular, any coordinate chart (U, ϕ) *around* x *gives an isomorphism* $T_xM \cong \mathbb{R}^m$.

PROOF. We first show that for any $v \in T_xM$, $v(f)$ depends only on the restriction of f to an arbitrary open neighborhood of x. Equivalently, we have to show that if f vanishes on a neighborhood of x then $v(f) = 0$. Using a coordinate chart around x construct $\chi \in C^\infty(M)$ with $\chi = 1$ on a neighborhood of x and $\chi = 0$ on a neighborhood of the support of f. Let $g = 1 - \chi$. Then $fg = f$ since $g = 1$ on $\mathrm{supp}(f)$. Since both f, g both vanish at x, $v(f) = v(fg) = 0$, as required.

This result can be re-interpreted as follows: Let $V \subset M$ be an open neighborhood of x, and $\mathcal{F}_V(M) \subset C^\infty(M)$ be the functions supported in V. Then T_xM can also be defined as the space of linear maps $\mathcal{F} \to \mathbb{R}$ satisfying the Leibnitz rule. Indeed, if χ is supported on V and $\chi = 1$ near x, then any function f coincides with $\chi f \in \mathcal{F}_V(M)$ near x. In particular, choose V with $\overline{V} \subset U$. Then $\mathcal{F}_V(U) = \mathcal{F}_V(M)$, and it follows directly that $T_xM = T_xU$. □

DEFINITION 1.21. Let $F \in C^\infty(N, M)$ be a smooth map, and $x \in N$. The tangent map

$$\mathrm{d}_xF : T_xN \to T_{F(x)}M$$

is defined as follows:

$$(\mathrm{d}_xF(v))(f) = v(F^*f), \quad f \in C^\infty(M).$$

It is immediate from the definition that d_xF is a linear map. We often write $F_* : T_xN \to T_{F(x)}M$ if the base point is understood. The map F_* is also called *push-forward*. Under composition of maps, $(F_1 \circ F_2)_* = (F_1)_* \circ (F_2)_*$.

EXERCISE 1.22. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open subsets and $F \in C^\infty(U, V)$. For all $x \in U$, the isomorphisms $T_xU = \mathbb{R}^m$ and $T_{F(x)}V = \mathbb{R}^n$ identify the tangent map $\mathrm{d}_xF : T_xU \to T_{F(x)}V$ with the Jacobian $D_xF : \mathbb{R}^m \to \mathbb{R}^n$. That is,

$$(\mathrm{d}_xF)(\frac{\partial}{\partial x^i}) = \sum_j \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j},$$

where x^i, y^j are the coordinates on U, V.

Thus d_xF is just the coordinate-free definition of the Jacobian: any choice of charts (U, ϕ) around x and (V, ψ) around $F(x)$ identifies d_xF with the Jacobian of the map $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ at $\phi(x)$. In particular,

$$\mathrm{rank}_x(F) = \mathrm{rank}(\mathrm{d}_xF).$$

F is an immersion if d_xF is injective everywhere, and a submersion if d_xF is surjective everywhere.

DEFINITION 1.23. A map $F \in C^\infty(N, M)$ is called an embedding if F is an injective immersion and F is a homeomorphism onto $F(N)$ (with the subspace topology).

Thus, a 1:1 immersion is an embedding if and only if the map $N \to F(N)$ is open for the subset topology. That is, one has to verify that for each open subset U of N, the image $F(U)$ can be written $F(U) = F(N) \cap V$ where V is open in M.

Example 1.24. Consider the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \ t \mapsto (\sin(2t), \cos(t)).$$

Then γ is an immersion, with image a "figure 8". Let F be the restriction of γ to the open interval $(-\pi/2, 3\pi/2)$. Then F is a 1-1 immersion, but is not an embedding. For instance, the image of the open interval $(0, \pi)$ is not open in the subspace topology. Note that the image of F is still the full figure 8, so it is not an embedded submanifold.

Exercise 1.25. Show that the image of an embedding is an embedded submanifold. Conversely, if $N \subset M$ is an embedded submanifold with the induced structure of a manifold on N, then the inclusion map $N \to M$ is an embedding.

If F is an embedding, then $\mathrm{d}_x F : T_x N \to T_{F(x)} M$ is injective, so $T_x N$ is identified with a subspace of $T_{F(x)} M$. In particular, if N is an embedded submanifold of $\mathbb{R}^m$, the tangent spaces to N are canonically identified with subspaces of $\mathbb{R}^m$.

Exercise 1.26. Suppose $F \in C^\infty(N, M)$ has $a \in M$ as a regular value, so $F^{-1}(a)$ is an embedded submanifold of N. Show that

$$T_x(F^{-1}(a)) = \ker(\mathrm{d}_x F) \subset T_x N$$

for all $x \in F^{-1}(a)$. (Hint: Use the normal form theorem for submersions.)

1.5. Velocity vectors for curves. If $I \subset \mathbb{R}$ is an open interval, a map $\gamma \in C^\infty(I, M)$ is called a *parametrized curve in M*. For any $t \in I$, one can define the velocity vector

$$\dot{\gamma}(t) \in T_{\gamma(t)} M$$

by $\dot{\gamma}(t) = (\gamma)_*(\frac{\partial}{\partial t})$. The action of the velocity vector on functions is, by definition of push-forward,

$$\dot{\gamma}(t)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)).$$

Exercise 1.27. Show that every $v \in T_x M$ is of the form $v = \dot{\gamma}(0)$ for some curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$. (Hint: Use a chart.)

Exercise 1.28. Let $F \in C^\infty(N, M)$ and $\gamma \in C^\infty(I, N)$ a smooth curve. Show that

$$F_*(\dot{\gamma}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)),$$

the tangent vector for the curve $F \circ \gamma$.

1.6. Jet spaces. Let $C_x^\infty(M)$ be the space of functions vanishing at x, and $C_x^\infty(M)^2 \subset C_x^\infty(M)$ linear combinations of products of such functions. Thus $C_x^\infty(M)^2$ consists of functions on M that vanish *to second order* at x. From the definition, it is clear that any tangent vector is determined by its value on $C_x^\infty(M)$, and is zero on $C_x^\infty(M)^2$. Thus one has a natural linear map $T_xM \to (C_x^\infty(M)/C_x^\infty(M)^2)^*$.

PROPOSITION 1.29. *The map*

$$T_xM \cong (C_x^\infty(M)/C_x^\infty(M)^2)^*$$

is a vector space isomorphism.

PROOF. Clearly, this map is injective. To show surjectivity, we have to show that if $v : C^\infty(M) \to \mathbb{R}$ is any linear map vanishing on constants and on $C_x^\infty(M)^2$, then v satisfies the Leibniz rule. Given f, g write $f = a + (f - a)$ where $f(x) = a$, and $g = b + (g - b)$ where $g(x) = b$. Then $f - a$ and $g - b$ vanish at x. Thus

$$\begin{aligned} v(fg) &= v(ab) + av(g-b) + v(f-a)b + v((f-a)(g-b)) \\ &= av(g-b) + v(f-a)b \\ &= av(g) + v(f)b. \end{aligned}$$

□

EXERCISE 1.30. Show that $C_x^\infty(M)$ is a maximal ideal in the algebra $C^\infty(M)$, and that every maximal ideal is of this form, for some x.

EXERCISE 1.31. Give the definition of the tangent map d_xF from this point of view.

This second definition suggests a definition of "higher tangent bundles": One defines, for all $k = 1, 2, \ldots$

$$J_x^kM := C^\infty(M)/C_x^\infty(M)^k,$$

the kth *jet space*. For any function $f \in C_x^\infty(M)$, its image $J_x^k(f)$ in J_x^kM is called the kth order jet at x. In local coordinates, $J_x^k(f)$ represents the kth order Taylor expansion at x.

Thus $J_x^0M = \mathbb{R}$, $J_x^1M = \mathbb{R} \oplus T_x^*M$. For larger k one still has projection maps $J_x^{k+1}M \to J_x^kM$ but these maps no longer split: There is no coordinate invariant meaning of the terms of order exactly $k+1$ of a Taylor expansion, unless the terms of order $\le k$ vanish.

2. Partitions of unity

Before developing the theory of vector fields, differential forms etc., we need an important technical tool known as *partitions of unity*.

We recall a few notions from topology. An open cover $(V_\beta)_{\beta \in B}$ of a topological space M is called a *refinement* of a given open cover $(U_\alpha)_{\alpha \in A}$ if each V_β is contained in some U_α. A cover $(U_\alpha)_{\alpha \in A}$ is called locally finite if each point in M has an open neighborhood that intersects only finitely many U_α's. A topological space is called *paracompact* if every open cover has a locally finite refinement. We will show now that manifolds are paracompact. First we need:

LEMMA 2.1. *Every manifold has an open covering* $(S_i)_{i=1}^N$ *where each* $\overline{S}_i$ *is compact, and* $\overline{S}_i \subset S_{i+1}$*, for all* i.

PROOF. Let $(U_i)_{i=1}^{\infty}$ be a countable basis of the topology. Already the U_i's with compact closure are a basis for the topology, so passing to a subsequence we may assume that each U_i has compact closure. Let $S_1 = U_1$. Let $k(1) > 1$ be an integer such that $U_1, \ldots, U_{k(1)}$ cover $\overline{S}_1$. Put $S_2 := U_1 \cup \ldots \cup U_{k(1)}$. Let $k(2) > k(1)$ be an integer such that $U_1, \ldots, U_{k(2)}$ cover $\overline{S}_2$. Put $S_3 := U_1 \cup \ldots \cup U_{k(2)}$. Proceeding in this fashion, one produces a sequence S_i with the required properties. □

THEOREM 2.2. *Every manifold* M *is paracompact. In fact, every open cover admits a countable, locally finite refinement consisting of open sets with compact closures.*

PROOF. Let $(U_\alpha)_{\alpha\in A}$ be any given cover of M. Here A is any indexing set. Let $S_1, S_2, \ldots$ be the sequence from Lemma 2.1. Each $\overline{S}_i$ is compact, and is therefore covered by finitely many U_α's. It follows that there exists a countable subset $A' \subset A$ such that $(U_\alpha)_{\alpha\in A'}$ is a covering of M. Replacing A with A', we may assume that our indexing set is $A = \{1, 2, \ldots\}$ (possibly finite). For each j, let $k(j)$ be an integer such that the sets U_i with $i \le k(1)$ cover $\overline{S}_j$. For $1 \le k \le k(j)$ define

$$V_k^j = U_k \cap (S_{j+1} \backslash \overline{S}_{j-1})$$

where j is the integer such that $k(j-1) < k \le k(j)$ (we put $k(0) = 0$ and $S_0 = \emptyset$). Then the collection V_k^j's are a locally finite refinement of the given cover, and each V_k^j has compact closure. □

LEMMA 2.3. *Let* $C \subset M$ *be a compact subset of some manifolds* M*. For any open neighborhood* U *of* C *there exists a smooth function* $f \in C^\infty(M)$ *with* $0 \le f \le 1$*, such that* $f = 1$ *on* C *and* $\operatorname{supp}(f) \subset U$.

PROOF. For each $x \in C$, choose a function $f_x \in C^\infty(M)$ with $\operatorname{supp}(f_x) \subset U$ and $f_x = 1$ on a neighborhood of x. (Such a function is easily constructed using a local coordinate chart.)Let $C_x = f_x^{-1}(1)$. Then $\{\operatorname{int}(C_x)\}_{x\in C}$ are an open cover of C. By compactness, there exists a finite subcover. Thus we can choose a finite collection of points x_i such that the interiors of the sets $C_i = C_{x_i}$ cover C. Write $f_i = f_{x_i}$. Then

$$f = 1 - \prod_{i=1}^{N} (1 - f_i)$$

has all the required properties. □

LEMMA 2.4 (Shrinking Lemma). *Let* $(U_\alpha)_{\alpha\in A}$ *be a locally finite covering of a manifold* M *by open sets with compact closure. Then there exists a covering* $(V_\alpha)_{\alpha\in A}$ *such that* $\overline{V_\alpha} \subset U_\alpha$ *for all* $\alpha \in A$.

Proof. Since the covering is locally finite, it is in particular countable. (Taking an open cover S_i of M as above, where each $\overline{S}_i$ is compact and contained in S_{i+1}, only finitely many U_α's meet each S_i. This implies countability.) Thus we may assume that $A = \mathbb{N}$ or a finite sequence, and write the given cover as $(U_i)_{i=1}^N$, $N \in \mathbb{N} \cup \infty$. Inductively, we now construct open subsets V_i with $\overline{V_i} \subset U_i$, such that $V_1, \ldots, V_i, U_{i+1}, U_{i+2}, \ldots$ still form a cover of M. Given $V_1, \ldots, V_i$, we construct V_{i+1} as follows: Choose $0 \le f \le 1$ supported in U_{i+1} with $f = 1$ on the compact set

$$C = M \backslash \big(\bigcup_{k \le i} V_i \cup \bigcup_{k \ge i+2} U_i \big) \subseteq U_{i+1}.$$

Then take V_{i+1} to be the open set where $f > 0$. Then $V_1, \ldots, V_{i+1}, U_{i+2}, \ldots$ is an open cover. Since the original cover was locally finite, $(V_i)_{i=1}^N$ is a cover of M. □

Theorem 2.5. *Let $(U_\alpha)_{\alpha \in A}$ be an open covering of a manifold M. Then there exists a* partition of unity subordinate to the cover U_α, *that is, a collection of functions χ_α such that*

(a) *Each point $x \in M$ has an open neighborhood U meeting the support of only finitely many χ_α's.*
(b) $0 \le \chi_\alpha \le 1$,
(c) $\mathrm{supp}(\chi_\alpha) \subset U_\alpha$,
(d) $\sum_\alpha \chi_\alpha = 1$

(The sum is well-defined, since near each point only finitely many χ_α are non-zero.)

Proof. Suppose first that the cover is locally finite and that each $\overline{U}_\alpha$ is compact. Choose a shrinking $(V_\alpha)_{\alpha \in A}$ as in the Lemma. Choose functions $0 \le f_\alpha \le 1$ supported in U_α with $f_\alpha = 1$ on V_α. Since the covering is locally finite, the sum $f = \sum_\alpha f_\alpha$ exists, and clearly $f > 0$ everywhere. Put $\chi_\alpha = f_\alpha / f$. This proves the Theorem for locally finite covers with compact closures. In the general case, choose a locally finite refinement $(\tilde{U}_\beta)_{\beta \in B}$ consisting of open sets with compact closures. There is a function $j : B \to A$, $\beta \mapsto \alpha = j(\beta)$ such that $\tilde{U}_\beta \subseteq U_{j(\beta)}$, and define $\chi_\alpha = \sum_{j(\beta)=\alpha} \tilde{\chi}_\beta$. □

Exercise 2.6. Show that Lemma 2.3 holds for any closed, not necessarily compact, subset of M.

Here is a typical application of partitions of unity.

Theorem 2.7. *Every manifold M can be embedded into R^k, for k sufficiently large.*

Proof. We will prove this only under the additional assumption that M admits a finite atlas. The existence of a finite atlas is obvious if M is compact. It can be shown that in fact, *every* manifold admits a finite atlas but the proof is not so easy (see e.g. the book Greub-Halperin-Vanstone). Let $(U_k, \phi_k)_{k=1}^N$ be a finite atlas. Choose a partition of unity subordinate χ_k, subordinate to the cover U_k. Then $\chi_k \phi_k : U_k \to \mathbb{R}^m$ extends by zero to a function $f_k \in C^\infty(M, \mathbb{R}^m)$. Let

$$F : M \to \mathbb{R}^{(m+1)N},\ x \mapsto (f_1(x), \ldots, f_N(x), \chi_1(x), \ldots, \chi_N(x)).$$

We claim that F is an embedding: That is, F is a 1-1 immersion that is a homeomorphism onto its image. First, F is 1-1. For suppose $F(x) = F(y)$. Thus $\chi_i(x) = \chi_i(y)$ for all i. Choose i with $\chi_i(x) > 0$. In particular, $x, y \in U_i$. Dividing the equation $f_i(x) = f_i(y)$ by $\chi_i(x) = \chi_i(y)$ we find $\phi_i(x) = \phi_i(y)$, thus $x = y$. More generally, for any $x \in N$ and any sequence $y_i \in N$, $F(y_i) \to F(x)$ implies $y_i \to x$. This shows that the inverse map is continuous, so F is a homeomorphism onto its image. Finally, let us show that F has maximal rank at x: For suppose $v \in \ker(\mathrm{d}_x F)$. Then $v(\chi_i) = 0$ and $v(f_i) = 0$ for all i. Choose i such that $\chi_i(x) > 0$, thus $x \in U_i$. Writing $f_i = \chi_i \phi_i$ the product rule gives

$$0 = v(\phi_i \chi_i) = \chi_i(x) v(\phi_i),$$

thus $v(\phi_i) = 0$. But ϕ_i is a diffeomorphism onto its image, so $v = 0$. □

A famous theorem of Whitney (1944) says that any manifold of dimension $\dim M = m$ can be embedded into $\mathbb{R}^{2m}$, and immersed into $\mathbb{R}^{2m-1}$. See Smale, Bull.Am.Math.Soc. 69 (1963), 133-145 for a survey of results on embeddings and immersions.

3. Vector fields

3.1. Vector fields. Suppose $\mathcal{A}$ is an algebra over $\mathbb{R}$. A *derivation* of $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ satisfying the Leibnitz rule

$$D(ab) = aD(b) + D(a)b.$$

If D_1, D_2 are derivations, then so is their commutator $[D_1, D_2] = D_1 D_2 - D_2 D_1$. Recall that the commutator satisfies the Jacobi identity,

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0.$$

Thus $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra of the algebra $\mathrm{End}(\mathcal{A})$. If $\mathcal{A}$ is commutative, the space $\mathrm{Der}(\mathcal{A})$ is an $\mathcal{A}$-submodule of $\mathrm{End}(\mathcal{A})$: If D is a derivation and $x \in \mathcal{A}$, then xD is also a derivation.

Definition 3.1. A vector field on M is a derivation $X \in \mathrm{Der}(C^\infty(M))$. That is, X is a linear map $X : C^\infty(M) \to C^\infty(M)$ satisfying the Leibnitz rule

$$X(fg) = (Xf)g + f(Xg), \quad f, g \in C^\infty(M).$$

The space of vector fields will be denoted $\mathfrak{X}(M)$. If $X, Y \in \mathfrak{X}(M)$, the vector field $[X, Y] = X \circ Y - Y \circ X$ is called the *Lie bracket* of X and Y.

Thus, the space $\mathfrak{X}(M)$ of vector fields is a Lie algebra. They are also a $C^\infty(M)$-module, that is, $fX \in \mathfrak{X}(M)$ for all $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Evaluation at any point $x \in M$ defines a linear map

$$\mathfrak{X} \to T_x M, \quad X \mapsto X_x, \quad X_x(f) = (Xf)(x).$$

Conversely X can be recovered from the collection of tangent vectors X_x with $x \in M$. We write

$$\mathrm{supp}(X) = \overline{\{x \in M |\, X_x \neq 0\}}.$$

One can think of a vector field as a family of tangent vectors depending smoothly on the base point.

Exercise 3.2 (Locality). Show that vector fields are local, in the following sense: If U is an open subset of a manifold M, and $X \in \mathfrak{X}(M)$, there exists a unique vector field $X_U \in \mathfrak{X}(U)$ (called the restriction of X to U) such that $(X_U)_x = X_x$ for all $x \in U$.

The action of X on functions $f \in C^\infty(M)$ supported in U is given in terms of the restriction by $Xf|_U = X_U(f|_U)$. Using a partition of unity, this means that the restrictions of X to coordinate charts determine the vector field X. In local coordinates, vector field have the following form:

Lemma 3.3. *Suppose $U \subset \mathbb{R}^n$ is an open subset. Then any $X \in \mathfrak{X}(U)$ has the form*

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$$

where $a_i \in C^\infty(U)$.

Proof. From the description of tangent vectors $v \in T_x U$, we know that

$$X_x = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$$

for some function $a_i(x)$. We have to show that a_i is smooth. But this is clear since $a_i = X(f)$ where $f \in C^\infty(U)$ is the coordinate function $f(x) = x^i$. □

Any *diffeomorphism* $F \in C^\infty(N, M)$ induces a map $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ of vector fields, with $F^*(F_*(X)(f)) = X(F^* f)$. Thus

$$F_*(X_x) = (F_* X)_{F(x)}.$$

For a general map F, this equation does need not define a vector field $F_* X$, for two reasons: 1) If F is not surjective, the equation does not specify $F_* X$ at points away from the image of F, 2) If F is not injective, the equation assigns more than one value to points $y \in M$ having more than one pre-image.

Definition 3.4. Let $F \in C^\infty(N, M)$. Two vector fields $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ are called F-related if

$$F^*(Y(f)) = X(F^* f)$$

for all $f \in C^\infty(M)$. One writes $X \sim_F Y$.

Proposition 3.5. *$X \sim_F Y$ if and only if*

$$F_*(X_x) = Y_{F(x)}$$

for all $x \in N$. If $X_1 \sim_F Y_1$ and $X_2 \sim_F Y_2$ then

$$[X_1, X_2] \sim_F [Y_1, Y_2].$$

Proof. Exercise. □

For example, if F is the embedding of a submanifold, then $X \in \mathfrak{X}(N)$ is F-related to a vector field $Y \in \mathfrak{X}(M)$ if and only if Y is tangent to $F(N)$ everywhere. The vector field X is then just the restriction of Y to N.

DEFINITION 3.6. A *Riemannian metric* on a manifold M is a symmetric $C^\infty(M)$-bilinear form $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ which is positive definite in the sense that for all $x \in M$, $g(X,X)(x) > 0$ for $X_x \neq 0$.

EXERCISE 3.7. Every manifold M admits a Riemannian metric g. (Hint: Define such a metric in charts, and then use a partition of unity to patch the local solutions together.) If g is a Riemannian metric and $x \in M$, the value $g(X,X)(x)$ depends only on X_x. If $U \subset M$ is open, there exists a unique Riemannian metric g_U such that $g_U(X_U, Y_U) = g(X,Y)|_U$.

3.2. The flow of a vector field. Suppose $\mathcal{A}$ is a finite dimensional algebra, and $\mathrm{Aut}(\mathcal{A})$ its group of algebra automorphisms. A 1-parameter subgroup of automorphisms is a group homomorphism $\Phi : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}), t \mapsto \Phi_t$, i.e. a map such that $\Phi_{t_1+t_2} = \Phi_{t_1}\Phi_{t_2}$ for all $t_1, t_2 \in \mathbb{R}$. For any 1-parameter group of automorphisms, the derivative $D = \frac{d}{dt}|_{t=0}\Phi_t : \mathcal{A} \to \mathcal{A}$ is well defined. By taking the derivative of the equation

$$\Phi_t(ab) = \Phi_t(a)\Phi_t(b)$$

one finds that $D \in \mathrm{Der}(\mathcal{A})$. Conversely, every derivation gives ride to a 1-parameter group $\Phi_t = \exp(tD)$ (exponential of matrices), and one check that each Φ_t is an automorphism. Thus, if $\dim \mathcal{A} < \infty$ one can identify derivations of an algebra with 1-parameter groups of automorphisms.

For the infinite-dimensional algebra $\mathcal{A} = C^\infty(M)$, this does not directly apply. We will see that so-called *complete* vector fields $X \in \mathfrak{X}(M)$ define a 1-parameter group of diffeomorphisms Φ_t of M, where the algebra automorphism Φ_t^* of $C^\infty(M)$ plays the role of $\exp(tX)$. This "flow" of X is constructed using integral curves.

Let X be a vector field on a manifold M. A curve $\gamma : I \to M$ is called an *integral curve* of X if for all $t \in I$,

$$\dot\gamma(t) = X_{\gamma(t)}.$$

If $U \subset \mathbb{R}^m$ is an open subset and $X \in \mathfrak{X}(U)$ is a vector field on U, this has the following interpretation. Letting x^i be the local coordinates on $\mathbb{R}^m$, write

$$X = \sum_i a^i \frac{\partial}{\partial x^i}$$

where $a_i \in C^\infty(U)$. Also write $\gamma(t) = (\gamma^1(t), \ldots, \gamma^m(t))$. Then the tangent vector $\dot\gamma(t) \in T_{\gamma(t)}U$ is just

$$\dot\gamma(t) = \sum_i \dot\gamma^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}.$$

To see this, apply $\dot\gamma(t)$ to a function $f \in C^\infty(U)$:

$$\dot\gamma(t)(f) = \frac{\partial}{\partial t} f(\gamma(t)) = \sum_i \frac{d\gamma^i}{dt} \frac{\partial f}{\partial x^i}\Big|_{\gamma(t)}.$$

Thus γ is an integral curve if and only its components satisfy the following system of first order ordinary differential equation:

$$\frac{d\gamma^i}{dt} = a^i(\gamma(t))$$

From ODE theory, it follows that for any $x \in U$, there exists a unique maximal solution γ_x of this system: That is, there is an open interval I_x around $t = 0$, and a curve $\gamma_x : I_x \to U$ with

$$\gamma_x(0) = x, \ \ \dot\gamma_x^i(t) = a^i(\gamma_x(t))$$

such that any other solution of this initial value problem is obtained by restriction to a subinterval. Moreover, this solution depends smoothly on the initial value x. By applying this result in manifold charts, one obtains:

Theorem 3.8. *Let X be a vector field on a manifold M. For each $x \in M$ there exists a unique maximal solution $\gamma_x : I_x \to M$ to the initial value problem*

$$\gamma_x(0) = x, \ \dot\gamma_x(t) = X_{\gamma_x(t)}.$$

The solution depends smoothly on the initial value x, in the following sense:

$$U = \bigcup_{x\in M} (\{x\} \times I_x) \subset M \times \mathbb{R}.$$

Then U is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$, and the map

$$\Phi : U \to M, \ (x, t) \mapsto \gamma_x(t)$$

is smooth.

Here "maximal solution" means that any other solution is obtained by restriction to some subinterval.

One calls Φ the *flow* of the vector field X. If $U = M \times \mathbb{R}$, that is if $I_x = \mathbb{R}$ for all x, the vector field is called *complete*, and the flow is called a *global flow*. A simple example for an incomplete vector field is $X = \frac{\partial}{\partial t}$ on $M = (0, 1) \subset \mathbb{R}$. By choosing a diffeomorphism $(0, 1) \cong \mathbb{R}$, one obtains an incomplete vector field on $\mathbb{R}$. Let us write $\Phi_t(x) = \Phi(x, t)$ for $t \in I_x$.

Exercise 3.9. Show that on a compact manifold, any vector field is complete. More generally, any compactly supported vector field on a manifold is complete.

Theorem 3.10. *Suppose $X \in \mathfrak{X}(M)$ is a complete vector field, and Φ_t its flow. Then each Φ_t is a diffeomorphism, and the map*

$$\mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \Phi_t$$

is a group homomorphism. In particular,

$$\Phi_0 = \mathrm{Id}, \quad \Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}.$$

Conversely, if $t \mapsto \Phi_t$ is any such group homomorphism and if the map $\Phi(t,x) = \Phi_t(x)$ is smooth, then Φ_t is the flow of a uniquely defined complete vector field X, called the generating vector field for the flow. For all $f \in C^\infty(M)$ one has

$$X(f) = L_X(f) := \frac{d}{dt}_{t=0} \Phi_t^* f.$$

Proof. We first prove the identity $\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}$ for $t_1, t_2 \in \mathbb{R}$. Let $x \in M$. Given t_2, both $\gamma(s) = \Phi_s(\Phi_{t_2}(x))$ and $\tilde{\gamma}(s) = \Phi_{s+t_2}(x)$ are integral curves for X. Indeed,

$$\frac{d}{ds}\tilde{\gamma}(s) = \frac{d}{du}\Phi_u(x)|_{u=s+t_2} = X_{\Phi_u(x)}|_{u=s+t_2} = X_{\tilde{\gamma}(s)}.$$

Since $\gamma, \tilde{\gamma}$ have the same initial value, they coincide on their domain of definition. In particular, $\gamma(t_1) = \tilde{\gamma}(t_1)$ which proves $\Phi_{t_1}(\Phi_{t_2}(x)) = \Phi_{t_1+t_2}(x)$. In particular, if X is complete, this equation holds for all t_1, t_2. Setting $t_1 = t, t_2 = -t$ we see that Φ_{-t} is a smooth inverse to Φ_t. Conversely, if Φ_t is a global flow, define X by $X_x = \frac{\partial}{\partial t}|_{t=0}\Phi_t(x)$. Using local coordinates, one checks that this defines a smooth vector field. □

Exercise 3.11. How much of this theorem goes though for incomplete vector fields?

Exercise 3.12. Show that the map $\Phi_t : \mathbb{R}^m \to \mathbb{R}^m$ given by multiplication by e^t is a global flow, and give a formula for the generating vector field. More generally, if A is any $m \times m$ matrix, show that the map

$$\Phi_t(x) = e^{tA}x$$

is a global flow, and find its generating vector field.

Exercise 3.13. Let $X \in \mathfrak{X}(N)$, $Y \in \mathfrak{X}(M)$ be vector fields and $F \in C^\infty(N, M)$ a smooth map. Show that $X \sim_F Y$ if and only if it intertwines the flows Φ_t^X, Φ_t^Y: That is,

$$F \circ \Phi_t^X = \Phi_t^Y \circ F.$$

Exercise 3.14. Show that for any vector field $X \in \mathfrak{X}(M)$ and any $x \in M$ with $X_x \neq 0$, there exists a local chart around x in which X is given by the constant vector field $\frac{\partial}{\partial x^1}$. Hint: Show that if S is an embedded codimension 1 submanifold, with $x \in S$ and $X_x \notin T_xS$, the map $U \times (-\epsilon, \epsilon) \to M$ is a diffeomorphisms onto its image, fo some open neighborhood U of x in S. Use the time parameter t and a chart around $x \in U$ to define a chart near x.

For any vector field $X \in \mathfrak{X}(M)$ and any diffeomorphism $F \in C^\infty(N, M)$, we define $F^*X \in \mathfrak{X}(N)$ by

$$F^*X(F^*f) = F^*(X(f)).$$

Thus $F^* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is just the inverse map to $F_* : \mathfrak{X}(N) \to \mathfrak{X}(M)$. Any complete vector field $X \in \mathfrak{X}(M)$ with flow Φ_t gives rise to a map $\Phi_t^* : \mathfrak{X}(M) \to \mathfrak{X}(M)$. One defines the *Lie derivative* L_X of a vector field $Y \in \mathfrak{X}(M)$ by

$$L_X(Y) = \frac{d}{dt}\Big|_{t=0} \Phi_t^* Y \in \mathfrak{X}(M).$$

In fact, this definition also makes sense for incomplete X: (To define the restriction of $L_X Y$ to an open set $U \subset M$ with compact closure, let $\epsilon > 0$ be small enough such $[-\epsilon, \epsilon] \subset I_x$ for all $x \in U$. Then $\Phi_t : U \mapsto \Phi_t(U)$ is defined for $|t| < \epsilon$, and the equation above makes sense.)

THEOREM 3.15. *For any $X, Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y$ is just the Lie bracket $[X, Y]$. One has the identity*

$$[L_X, L_Y] = L_{[X,Y]}.$$

PROOF. Let $\Phi_t = \Phi_t^X$ be the flow of X. For $f \in C^\infty(M)$ we calculate,

$$\begin{aligned} (L_X Y)(f) &= \frac{d}{dt}|_{t=0}(\Phi_t^* Y)(f) \\ &= \frac{d}{dt}|_{t=0}\Phi_t^*(Y(\Phi_{-t}^*(f))) \\ &= \frac{d}{dt}|_{t=0}(\Phi_t^*(Y(f)) - Y(\Phi_t^*(f)) \\ &= X(Y(f)) - Y(X(f)) \\ &= [X, Y](f). \end{aligned}$$

The identity $[L_X, L_Y] = L_{[X,Y]}$ just rephrases the Jacobi identity for the Lie bracket. □

The definition of Lie derivative gives the formula

$$\frac{d}{dt}(\Phi_t^X)^* Y = (\Phi_t^X)^*(L_X Y) \tag{1}$$

by the calculation, $\frac{d}{dt}(\Phi_t^X)^* Y = \frac{d}{du}|_{u=0}(\Phi_{u+t}^X)^* Y = (\Phi_t^X)^* \frac{d}{du}|_{u=0}(\Phi_u^X)^* Y$. From this we obtain:

THEOREM 3.16. *Let X, Y be two complete vector fields, with flows Φ_t^X and Φ_s^Y. Then $[X, Y] = 0$ if and only if Φ_t^X and Φ_s^Y commute for all t, s: $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$.*

PROOF. Suppose $[X, Y] = 0$. Then

$$\frac{d}{dt}(\Phi_t^X)^* Y = (\Phi_t^X)^* L_X Y = (\Phi_t^X)^*[X, Y] = 0$$

for all t. Hence $(\Phi_t^X)^* Y = Y$ for all t, which means that Y is Φ_t^X-related to itself. It follows that Φ_t^X takes the flow Φ_s^Y of Y to itself, which is just the desired equation $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$. Conversely, by differentiating the equation $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$ with respect to s, t, we find that $[X, Y] = 0$. □

Thus $[X, Y]$ measures the extent to which Φ_t^X and Φ_t^Y fail to commute. This can be made more precise:

EXERCISE 3.17. Suppose X, Y are complete vector fields, and $f \in C^\infty(M)$.

a) Prove the formula for the kth order Taylor expansion of $(\Phi_t^X)^* f$:

$$(\Phi_t^X)^* f = \sum_{j=0}^{k} \frac{t^j}{j!}(X)^j(f) + O(t^{k+1}).$$

Formally, one writes $(\Phi_t^X)^* = \exp(tX)$.

b) Let F_t be the family of diffeomorphisms,

$$F_t = \Phi_{-t}^X \circ \Phi_{-t}^Y \circ \Phi_t^X \circ \Phi_t^Y.$$

Show that

$$F_t^* f = f + t^2\,[X, Y](f) + O(t^3).$$

4. Differential forms

4.1. Super-algebras. A *super-vector space* is a vector space E with a $\mathbb{Z}_2$-grading. Elements of degree $0 \mod 2$ are called *even*, and elements of degree $1 \mod 2$ are called *odd*. Thus a super-vector space is simply a vector space with a decomposition $E = E^0 \oplus E^1$. Elements in E^0 are called even and those in E^1 are called odd. We will also write $E^0 = E^{\mathrm{even}}$ and $E^1 = E^{\mathrm{odd}}$, in particularly if E carries other gradings that might be confused with the $\mathbb{Z}_2$-grading. The space $\mathrm{End}(E)$ of endomorphisms has a splitting

$$\mathrm{End}(E) = \mathrm{End}(E)^0 \oplus \mathrm{End}(E)^1,$$

where $\mathrm{End}(E)^0$ consists of endomorphisms preserving the splitting $E = E^0 \oplus E^1$, and $\mathrm{End}(E)^1$ consists of endomorphisms taking E^0 to E^1 and E^1 to E^0. $\mathcal{A} = \mathrm{End}(E)$ is a first example of a *super-algebra*, that is, a $\mathbb{Z}_2$-graded algebra [1] $\mathcal{A} = \mathcal{A}^{\mathrm{even}} \oplus \mathcal{A}^{\mathrm{odd}}$ such that

$$\mathcal{A}^k \mathcal{A}^l \subseteq \mathcal{A}^{k+l \bmod 2}$$

for $k, l \in \{0, 1\}$. The sign conventions of supermathematics decrees:

> **Super-sign convention:** A minus sign appears whenever two odd elements interchange their position.

Physicists thinks of elements of odd degree as "fermions", and those of even degree as "bosons". Her are some examples of the super-sign convention. The *tensor product of super-algebras* $\mathcal{A}, \mathcal{B}$ is defined to be the usual tensor product $\mathcal{A} \otimes \mathcal{B}$, with $\mathbb{Z}_2$-grading

$$(\mathcal{A} \otimes \mathcal{B})^k = \bigoplus_{l+m=k \bmod 2} \mathcal{A}^l \otimes \mathcal{B}^m,$$

and algebra structure

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{m_1 l_2}(a_1 a_2) \otimes (b_1 b_2)$$

[1] In this course, "algebra" always means an associative algebra over $\mathbb{R}$ with unit 1.

for $a_i \in \mathcal{A}^{l_i}$, $b_j \in \mathcal{B}^{m_j}$. Also, if $\mathcal{A}$ is any super-algebra, the *super-commutator* of two elements $a_1 \in \mathcal{A}^{k_1}$ and $a_2 \in \mathcal{A}^{k_2}$ is defined by

$$[a_1, a_2] = a_1 a_2 - (-1)^{k_1 k_2} a_2 a_1 \in \mathcal{A}^{k_1+k_2}.$$

PROPOSITION 4.1. *The super-commutator is super-skew symmetric,*

$$[a_1, a_2] = -(-1)^{k_1 k_2}[a_2, a_1]$$

and satisfies the super-Jacobi identity,

$$[a_1, [a_2, a_3]] + (-1)^{k_1(k_2+k_3)}[a_2, [a_3, a_1]] + (-1)^{k_3(k_1+k_2)}[a_3, [a_1, a_2]] = 0.$$

PROOF. Exercise. □

More generally, a super-space E with bracket $[\cdot, \cdot]$ satisfying these identities is called a *super-Lie algebra.*

An *derivation* of a superalgebra $\mathcal{A}$ of degree $r \in \{0, 1\}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ such that

$$D(ab) = D(a)b + (-1)^{kr} a D(b)$$

for all $a \in \mathcal{A}^k$ and $b \in \mathcal{A}^l$. Note that any derivation is determined by its values on generators for the algebra, and that $D(1) = 0$ for any derivation D. We denote

$$\mathrm{Der}(\mathcal{A}) = \mathrm{Der}^0(\mathcal{A}) \oplus \mathrm{Der}^1(\mathcal{A})$$

the space of super-derivations.

EXERCISE 4.2. Show that $\mathrm{Der}(\mathcal{A})$, with bracket the super-commutator of endomorphisms, is super-Lie algebra. If $\mathcal{A}$ is super-commutative, show that $\mathrm{Der}(\mathcal{A})$ is also an $\mathcal{A}$-module (by multiplication from the left).

EXERCISE 4.3. Suppose $\mathcal{A}$ is a super-algebra. Show that the map $\epsilon : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ taking $a \in \mathcal{A}$ to the operator $\epsilon(a)$ of multiplication (from the left) by a is a homomorphism of super-algebras, i.e. it preserves products and degrees.

EXERCISE 4.4. Let E, F be super-vector spaces. Define the tensor product of two endomorphisms of finite degree, in such a way that it respects the sign convention of supermathematics. Show that with this definition,

$$\mathrm{End}(E \otimes F) = \mathrm{End}(E) \otimes \mathrm{End}(F).$$

Many super-vector spaces arise as $\mathbb{Z}$-graded algebras $E = \bigoplus_{k\in\mathbb{Z}} E^k$, by reducing the degree mod 2:

$$E^{\mathrm{even}} = \bigoplus_{k\in\mathbb{Z}} E^{2k}, \quad E^{\mathrm{odd}} = \bigoplus_{k\in\mathbb{Z}} E^{2k+1}.$$

In this case, we will call E a $\mathbb{Z}$-graded super-vector space. Ordinary vector spaces E can be viewed as super-vector spaces by putting $E^{\mathrm{even}} = E$, $E^{\mathrm{odd}} = 0$.

4.2. Exterior algebra. We now give our main example of a graded algebra. Let E be a finite dimensional real vector space. (The example to keep in mind is $E = T_xM$, the tangent space to a manifold. But other examples will appear as well.)

For any finite dimensional vector space E over $\mathbb{R}$, one defines $\wedge^0 E^* = \mathbb{R}$, $\wedge^1 E^* = E^*$, and more generally

$$\wedge^k E^* = \{\text{antisymmetric } k\text{-linear maps } \ \omega : \underbrace{E \times \cdots \times E}_{k \text{ times}} \to \mathbb{R}\}.$$

Thus $\omega \in \wedge^k E^*$ if it is linear in each argument and satisfies

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\omega(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in E$ and any permutation σ of the index set $1, \ldots, k$. Each $\wedge^k E^*$ is a finite dimensional vector space. If $e_1, \ldots, e_n$ is a basis for E, any ω is determined by its values on $e_{i_1}, \ldots, e_{i_k}$ for any $i_1 < \ldots < i_k$. Since the number of ordered k-element subsets of $\{1, \ldots, n\}$ is $\frac{n!}{k!(n-k)!}$, it follows that

$$\dim \wedge^k E^* = \frac{n!}{k!(n-k)!}$$

for $k \le n$, and $\wedge^k E^* = 0$ for $k > n$. Note $\dim \wedge^{n-k} E^* = \wedge^k E^*$. Non-zero elements of the 1-dimensional vector space $\wedge^n E^*$ are called *volume elements*. The direct sum of the vector space $\wedge^k E^*$ is denoted[2]

$$\wedge E^* = \bigoplus_{k=0}^{n} \wedge^k E^*,$$

its dimension is $\dim \wedge E^* = 2^n$, the number of subsets of $\{1, \ldots, n\}$.

The graded algebra structure on the vector space $\wedge E^*$ is the so-called called *wedge product*.

For $\omega_1 \in \wedge^{k_1} E^*$ and $\omega_2 \in \wedge^{k_2} E^*$ one defines $\omega_1 \wedge \omega_2 \in \wedge^k E^*$, $k = k_1 + k_2$ by anti-symmetrization

$$(\omega_1 \wedge \omega_2)(v_1 \ldots, v_k) = \frac{1}{k_1! k_2!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\omega_1(v_{\sigma(1)}, \ldots, v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \ldots, v_{\sigma(k)})$$

where σ runs over all permutations. The wedge product extends to all of $\wedge E^*$ by linearity.

For example, if $f_1, f_2 \in E^*$ then

$$(f_1 \wedge f_2)(v_1, v_2) = f_1(v_1)f_2(v_2) - f_1(v_2)f_2(v_1).$$

More generally, if $f_1, \ldots, f_k \in E^*$,

$$(f_1 \wedge \cdots \wedge f_k)(v_1, \ldots, v_k) = \det(f_i(v_j)). \tag{2}$$

The following properties of the wedge product are left as an exercise.

[2] In the infinite dimensional case $\dim E = \infty$, one defines the exterior algebra somewhat differently. Fortunately, we will only be concerned with finite dimensions.

PROPOSITION 4.5. (a) *The wedge product is associative and super-commutative.*

(b) *Let* $e_1, \dots, e_n$ *be a basis of* E *and* $f^1, \dots, f^n$ *the dual basis of* E^**. For each ordered subset* $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ *let* $f^I = f^{i_1} \wedge \dots \wedge f^{i_k}$*. Then* $\{f^I | \#I = k\}$ *is a basis for* $\wedge^k E^*$*.*

EXERCISE 4.6. If E_1, E_2 are two finite dimensional vector space, show that

$$\wedge(E_1 \oplus E_2)^* = \wedge E_1^* \otimes \wedge E_2^*$$

(tensor product of graded algebras).

There are two important operations on $\wedge E^*$ called *exterior multiplication* and *contraction*. Exterior multiplication is just the algebra homomorphism

$$\epsilon : \wedge E^* \to \mathrm{End}(\wedge E^*)$$

for the graded commutative algebra $\wedge E^*$. Thus $\epsilon(\omega)(\beta) = \omega \wedge \beta$. For $v \in E$, one defines the *contraction operator*

$$\iota_v : \wedge^k E^* \to \wedge^{k-1} E^*$$

by $\iota_v(\omega)(v_2, \dots, v_k) = \omega(v, v_2, \dots, v_k)$. Sometimes we write $\iota_v = \iota(v)$.

Thus $\iota_v \in \mathrm{End}^{-1}(\wedge E^*)$. Clearly $\iota_v \circ \iota_v = 0$. Hence also

$$[\iota_v, \iota_w] = \iota_{v+w} \circ \iota_{v+w} = 0.$$

If $f_1, \dots, f_k$ are vectors in E^*,

$$\iota_v(f_1 \wedge \dots \wedge f_k) = \sum_j (-1)^{j+1} f_j(v) f_1 \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_k. \tag{3}$$

This follows from the definition of the wedge product, or equivalently by expanding the determinant on the right hand side of (2) at the first column.

THEOREM 4.7. *The contraction operator* ι_v *is a super-derivation of degree* -1 *of* $\wedge E^*$*. One has*

$$[\iota_v, \epsilon(\omega)] = \epsilon(\iota_v \omega).$$

PROOF. Let $f_1, \dots, f_r$ be vectors in E^*. From (3) we read off that for any $l \le r$,

$$\iota_v(f_1 \wedge \dots \wedge f_r) = \iota_v(f_1 \wedge \dots \wedge f_l) \wedge (f_l \wedge \dots \wedge f_r) + (-1)^l (f_1 \wedge \dots \wedge f_l) \wedge \iota_v(f_l \wedge \dots \wedge f_r).$$

By linearity, it follows that

$$\iota_v(\omega \wedge \beta) = \iota_v \omega \wedge \beta + (-1)^l \omega \wedge \iota_v \beta$$

for $\omega \in \wedge^l E^*$ and $\beta \in \wedge E^*$, proving that ι_v is a super-derivation. The identity $[\iota_v, \epsilon(\omega)] = \epsilon(\iota_v \omega)$ is just re-phrasing the same condition. □

For any linear map $A : F \to E$ the dual map $A^* : E^* \to F^*$ extends uniquely to an algebra homomorphism $A^* : \wedge E^* \to \wedge F^*$. Thus if $f_1, \dots, f_k \in E^*$,

$$A^*(f_1 \wedge \dots \wedge f_k) = (A^* f_1 \wedge \dots \wedge A^* f_k).$$

In particular, any endomorphism of E gives rise to an endomorphism of $\wedge E^*$. For any 1-parameter group $A_t = \exp(tL)$ of automorphisms, we obtain a 1-parameter group $A^*_{-t} = \exp(-tL)^*$ of automorphisms of $\wedge E^*$. It follows that $-L^* : E^* \to E^*$ extends to a derivation D_L of degree 0 of $\wedge E^*$. One has

$$D_L(f_1 \wedge \cdots \wedge f_k) = -\sum_j (-1)^{j+1} L^*(f_j) f_1 \wedge \cdots \widehat{f_j} \cdots \wedge f_k.$$

(Don't confuse D_L with the algebra homomorphism $(-L)^* : \wedge E^* \to \wedge E^*$!).

EXERCISE 4.8. a) Show that the map $\mathrm{End}(E) \to \mathrm{Der}^0(\wedge E^*)$, $L \mapsto D_L$ is a Lie algebra homomorphism.

b) For $v \in E$, $L \in \mathrm{End}(E)$, show that $[D_L, \iota_v] = -\iota_{L(v)}$. (Hint: Since both sides are derivations, it suffices to check on generators.)

After this lengthy discussion of linear algebra, let us return to manifolds.

4.3. Differential forms.

DEFINITION 4.9. For any manifold M and any $x \in M$, the dual space $T^*_x M := (T_x M)^*$ of the tangent space at x is called the *cotangent space* at x. The elements of $T^*_x M$ are called *covectors*. Elements of $\wedge^k T^* M$ are called k-forms on $T_x M$.

Any function $f \in C^\infty(M)$ determines a covector $\mathrm{d}_x f \in T^*_x M$, called its exterior differential at x, by

$$(\mathrm{d}f)_x(v) := v(f), \quad v \in T_x M.$$

If $U \subset \mathbb{R}^m$ is an open subset with coordinates $x^1, \ldots, x^m$ (viewed as functions on U), we can thus define 1-forms

$$(\mathrm{d}x^i)_x \in T^*_x U.$$

This 1-forms are the dual basis to the basis $\frac{\partial}{\partial x^i}\Big|_x$ of $T_x U$: Indeed,

$$(\mathrm{d}x^i)_x \left(\frac{\partial}{\partial x^j}\Big|_x \right) = \frac{\partial}{\partial x^j}(x^i)\Big|_x = \delta_{ij}.$$

A differential k-form on M is a family of k-forms on tangent spaces $T_x M$ depending smoothly on the base point, in the following sense:

DEFINITION 4.10. A differential k-form on M is a $C^\infty(M)$-linear map

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M)$$

that is anti-symmetric and $C^\infty(M)$-linear in each argument. The space of k-forms is denoted $\Omega^k(M)$, the direct sum of all $\Omega^k(M)$ is denoted $\Omega(M)$. In particular, $\Omega^0(M) = C^\infty(M)$.

For any $x \in M$ there is a natural evaluation map $\Omega^k(M) \to \wedge^k T_x^* M,\ \omega \mapsto \omega_x$ such that

$$\omega(X_1, \ldots, X_k)|_x = \omega_x((X_1)_x, \ldots, (X_k)_x).$$

Similarly, if $U \subset M$ is open, one can define the restriction $\omega|_U \in \Omega^k(U)$ such that

$$\omega(X_1, \ldots, X_k)|_U = \omega|_U((X_1)|_U, \ldots, (X_k)|_U).$$

If U, V are two open subsets, and ω_U, ω_V are forms on U and V with $\omega_U|_{U\cap V} = \omega_V|_{U\cap V}$ then there is a unique form $\omega_{U\cup V} \in \Omega(U \cup V)$ restricting to ω_U and ω_V, respectively.

One defines the wedge product on $\Omega(M)$ by anti-symmetrization, similar to the wedge product on $\wedge E^*$. It is uniquely defined by requiring that all the evaluation maps $\Omega(M) \to \wedge T_x^* M$ are algebra homomorphisms. Wedge product makes $\Omega(M)$ into a graded super-commutative super-algebra. For $X \in \mathfrak{X}(M)$, one denotes by $\iota_X = \iota(X) \in \mathrm{Der}^{-1}(\Omega(M))$ the operator of contraction by X, and for $\mu \in \Omega(M)$ one denotes by $\epsilon(\mu)$ the operator of left multiplication by μ. As before, we have $[\iota_X, \iota_Y] = 0$ and $[\iota_X, \epsilon(\mu)] = \epsilon(\iota_X \mu)$.

4.4. The exterior differential. The magic fact about the algebra of differential forms is the existence of a canonical derivation d of degree 1. For any function $f \in C^\infty(M)$ one defines $\mathrm{d}f \in \Omega^1(M)$ by the equation,

$$(\mathrm{d}f)(X) = X(f).$$

It has the property $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$. In particular, if $U \subset \mathbb{R}^m$ is an open subset, we have 1-forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^m \in \Omega^1(U)$. Writing $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ for any k-element subset $i_1 < \ldots < i_k$, the most general k-form on U reads

$$\omega = \sum_I \omega_I \mathrm{d}x^I$$

where $\omega^I \in C^\infty(U)$ are recovered as

$$\omega^I = \omega(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}).$$

THEOREM 4.11. *The map $d : \Omega^0(M) \to \Omega^1(M),\ f \mapsto df$ extends uniquely to a super-derivation of degree 1 on $\Omega(M)$, called* exterior differential*, with $d(df) = 0$. The exterior differential has property*

$$d \circ d = 0.$$

PROOF. It suffices to prove existence and uniqueness for the restrictions to elements of an open cover $(U_\alpha)_{\alpha\in A}$. Indeed, once we know that $\mathrm{d}\omega|_{U_\alpha}$ exist and are unique, then the forms $\mathrm{d}\omega|_{U_\alpha}$ agree on overlaps $U_\alpha \cap U_\beta$ by uniqueness, so they define a global form $\mathrm{d}\omega$. In particular, we may take $(U_\alpha)_{\alpha\in A}$ to be a covering by coordinate charts. This reduces the problem to open subsets $U \subset \mathbb{R}^m$.

The derivation property of d forces us to define $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ as follows,

$$\mathrm{d}(\sum_I \omega_I \mathrm{d}x^I) = \sum_I \mathrm{d}\omega_I \wedge \mathrm{d}x^I = \sum_j \sum_I \frac{\partial \omega^I}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^I.$$

We have

$$\mathrm{d}^2(\sum_I \omega_I \mathrm{d}x^I) = \sum_{jk} \sum_I \frac{\partial^2 \omega^I}{\partial x^k \partial x^j} \mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I = 0,$$

by equality of mixed partials. It remains to check d is a derivation. For $\omega = f\mathrm{d}x^I \in \Omega^k$ and $\nu = g\mathrm{d}x^J \in \Omega^l$ we have

$$\begin{aligned} \mathrm{d}(\omega \wedge \nu) &= \mathrm{d}(f\mathrm{d}x^I \wedge g\mathrm{d}x^J) \\ &= \mathrm{d}(fg) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\ &= (f\mathrm{d}g + g\mathrm{d}f) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\ &= \mathrm{d}f \wedge \mathrm{d}x^I \wedge g\mathrm{d}x^J + (-1)^k f \wedge \mathrm{d}x^I \wedge \mathrm{d}g \wedge \mathrm{d}x^J \\ &= (\mathrm{d}\omega) \wedge \nu + (-1)^k \omega \wedge \mathrm{d}\nu. \end{aligned}$$

□

Let us write out the exterior differential on $M = \mathbb{R}^3$, with coordinates x, y, z. On functions,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z,$$

so d is essentially the *gradient*. On 1-forms $\omega = f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z$,

$$\mathrm{d}\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\mathrm{d}x \wedge \mathrm{d}y + (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})\mathrm{d}y \wedge \mathrm{d}z + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})\mathrm{d}z \wedge \mathrm{d}x,$$

so d is essentially the *curl*, and on 2-forms $\omega = f\mathrm{d}y \wedge \mathrm{d}z + g\mathrm{d}z \wedge \mathrm{d}x + h\mathrm{d}x \wedge \mathrm{d}y$,

$$\mathrm{d}\omega = (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z})\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

so d is essentially the *divergence*. In general, one should think of d as the proper abstract setting for grad, curl, div.

4.5. Functoriality. Any smooth map $F \in C^\infty(N, M)$ between manifolds gives rise to an algebra homomorphism

$$F^* : \Omega(M) \to \Omega(N)$$

such that $(F^*\omega)_x = F^*(\omega_x)$.

THEOREM 4.12. *F^* commutes with exterior differential:*

$$F^* \circ d = d \circ F^*.$$

PROOF. The algebra $\Omega(M)$ is generated by all functions $f \in C^\infty(M)$ together with all differentials $\mathrm{d}f$. That is, any differential form can be written as a finite linear combination of expressions

$$f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k,$$

and finally by linearity it holds everywhere. This follows by choosing a finite atlas of M and a subordinate partition of unity. [3] Hence, it suffices to check the the identity on functions f and differentials $\mathrm{d}f$.

For $f \in C^\infty(M)$, and any $x \in M$, $v \in T_xM$, we have

$$(F^*(\mathrm{d}f))_x(v) = (\mathrm{d}f)_{F(x)}(\mathrm{d}_xF(v)) = (\mathrm{d}_xF(v))(f) = v(F^*f) = \mathrm{d}(F^*f)_x(v).$$

Thus $F^* \circ \mathrm{d} = \mathrm{d} \circ F^*$ on functions. On differentials $\mathrm{d}f$:

$$F^*(\mathrm{d}(\mathrm{d}f)) = 0, \ \mathrm{d}(F^*(\mathrm{d}f) = \mathrm{d}(\mathrm{d}(F^*f)) = 0.$$

□

For instance, if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open subsets with coordinates x^i, y^j, and $F \in C^\infty(U,V)$, with components $F^j = F^*y^j$, we have

$$F^*\mathrm{d}y^j = \mathrm{d}F^*y^j = \mathrm{d}F^j = \sum_i \frac{\partial F^j}{\partial x^i}\mathrm{d}x^i.$$

If $\dim U = \dim V = m$, we obtain in particular

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m) = \det(\frac{\partial F^j}{\partial x^i}) \ (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m).$$

Suppose $X \in \mathfrak{X}(M)$ is a complete vector field, with flow Φ_t. Then we have a 1-parameter group of automorphisms Φ_t^* of $\Omega(M)$. We define the Lie derivative to be its generator:

$$L_X\omega = \frac{d}{dt}|_{t=0}\Phi_t^*\omega.$$

The definition extends to incomplete vector fields: If U is any open subset with compact closure, the map $\Phi_t : U \to \mathcal{M}$ exists for $|t| < \epsilon$, and the equation defines $L_X\omega|_U$.

We have now defined three kinds of derivations of $\Omega(M)$: The contraction ι_X, the Lie derivative L_X, and the exterior differential d. Recall that the graded commutator of two derivations is again a derivation.

Theorem 4.13. *One has the following identities:*

$$\begin{aligned}
[d, d] &= 0,\\
[L_X, d] &= 0,\\
[\iota_X, d] &= L_X,\\
[\iota_X, \iota_Y] &= 0,\\
[L_X, L_Y] &= L_{[X,Y]},\\
[L_X, \iota_Y] &= \iota_{[X,Y]}.
\end{aligned}$$

[3] We mentioned earlier that any manifold admits a finite atlas, although we never proved this. The following proof can be easily modified to close this gap: it is enough to check the identity on functions supported in coordinate charts, where the f_i for $i > 0$ can be taken as coordinate functions.

PROOF. The first equation is just $[\mathrm{d},\mathrm{d}] = 2\mathrm{d}^2 = 0$. Since $(\Phi_X^t)^*$ and d commute, $[L_X, \mathrm{d}] = 0$ by definition of L_X. The other identities can be checked on generators f, $\mathrm{d}f$ of $\Omega(M)$ (using that two derivations are equal if and only if they agree on generators). For instance, the third equation is verified by the calculations,

$$[\iota_X, \mathrm{d}](f) = \iota_X \mathrm{d}f = (\mathrm{d}f)(X) = X(f) = L_X f$$

and

$$[\iota_X, \mathrm{d}](\mathrm{d}f) = \iota_X \mathrm{d}\mathrm{d}f + \mathrm{d}\iota_X \mathrm{d}f = \mathrm{d}L_X f = L_X \mathrm{d}f.$$

The fourth equation is obvious since both sides vanish on generators. The fifth equation, on functions f, is just the definition of the Lie bracket, and on $\mathrm{d}f$ follows since Lie derivatives and d commute. In the last equation, both sides vanish on functions, and on $\mathrm{d}f$ we have

$$[L_X, \iota_Y](\mathrm{d}f) = [[L_X, \iota_Y], \mathrm{d}](f) = [L_X, [\iota_Y, \mathrm{d}]](f) = [L_X, L_Y](f) = L_{[X,Y]}(f) = \iota_{[X,Y]}\mathrm{d}f.$$

(Alternatively, the last equation follows by direct application of the definition of the Lie derivative.) □

These identities are of fundamental importance in the Cartan's calculus of differential forms. It is somewhat remarkable that contractions, Lie derivatives and d form a graded Lie subalgebra of the graded Lie algebra of derivations!

PROPOSITION 4.14. *For any $\omega \in \Omega^k(M)$ and any $X_0, \ldots, X_k \in \mathfrak{X}(M)$,*

$$\begin{aligned}(d\omega)(X_0, \ldots, X_k) &= \sum_j (-1)^j X_j(\omega(X_0, \ldots, \widehat{X_j}, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)\end{aligned}$$

PROOF. We give the proof for $k = 2$, which will show the relevant pattern. The left hand side is $\iota_{X_2}\iota_{X_1}\iota_{X_0}\mathrm{d}\omega$. Using the commutation relations between d and contractions, we find

$$\begin{aligned}\iota_{X_2}\iota_{X_1}\iota_{X_0}\mathrm{d}\omega &= \iota_{X_2}\iota_{X_1}L_{X_0}\omega - \iota_{X_2}\iota_{X_1}\mathrm{d}\iota_{X_0}\omega \\ &= \iota_{X_2}\iota_{X_1}L_{X_0}\omega - \iota_{X_2}L_{X_1}\iota_{X_0}\omega + L_{X_2}\iota_{X_1}\iota_{X_0}\omega.\end{aligned}$$

The desired identity now follows by permuting the Lie derivatives in each term all the way to the left. For example, the first term gives

$$\begin{aligned}\iota_{X_2}\iota_{X_1}L_{X_0}\omega &= \iota_{X_2}L_{X_0}\iota_{X_1}\omega - \iota_{X_2}\iota_{[X_0,X_1]}\omega \\ &= L_{X_0}\iota_{X_2}\iota_{X_1}\omega - \iota_{[X_0,X_2]}\iota_{X_1}\omega - \iota_{X_2}\iota_{[X_0,X_1]}\omega.\end{aligned}$$

□

For example, if ω is a 1-form,

$$\mathrm{d}\omega(X_0, X_1) = X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]).$$

EXERCISE 4.15. Let U be an open subset of $\mathbb{R}^m$, and $X = \sum X_i \frac{\partial}{\partial x^i}$ a vector field. Let $\Gamma = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ be the standard "volume form". Show that

$$L_X \Gamma = (\sum_i \frac{\partial X_i}{\partial x^i})\ \Gamma.$$

Conclude that the flow of X is volume preserving if and only if the divergence $\mathrm{div}_\Gamma(X) = (\sum_i \frac{\partial X_i}{\partial x^i})$ vanishes everywhere.

4.6. Orientation of manifolds. A diffeomorphism $F : U \to V$ between open subsets of $\mathbb{R}^m$ is called orientation preserving if for all $x \in U$, the Jacobian $D_x F$ has positive determinant.

An atlas (U_α, ϕ_α) for a manifold M is said to be *oriented* if all transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ are orientation preserving. If such an atlas exists, M is called orientable. An orientable manifold M with an oriented atlas is called an oriented manifold. A Map $F \in C^\infty(N, M)$ between oriented manifolds of the same dimension are orientation preserving if its expressions in oriented charts are orientation preserving.

Orientability of manifolds M is closely related to the existence of *volume forms.* A volume form is an m-form $\Lambda \in \Omega^m(M)$, with $\Lambda_x \neq 0$ for all $x \in M$.

THEOREM 4.16. *A manifold M is orientable if and only if it admits a volume form. Any volume form Λ on a manifold M defines a unique orientation, with the property that for all oriented charts (U_α, ϕ_α),*

$$\phi_\alpha^*(dx^1 \wedge \cdots dx^n) = f_\alpha\, \Lambda|_{U_\alpha}$$

where $f_\alpha > 0$ everywhere. Two volume forms Λ, Λ' define the same orientation if and only if $\Lambda' = f\Lambda$ with $f > 0$.

PROOF. Suppose M is orientable, and let $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ be an oriented atlas. Choose a partition of unity χ_α subordinate to the cover $(U_\alpha)_{\alpha \in A}$. The form $\Lambda_\alpha = \phi_\alpha^*(\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^m)$ is a volume form on U_α. Set

$$\Lambda = \sum_\alpha \chi_\alpha\, \Lambda_\alpha.$$

Then Λ is a volume form on M. Indeed, if $v_1 \ldots, v_m$ is an oriented basis of $T_m M$, each of the forms $(\Lambda_\alpha)_x$ with $x \in U_\alpha$ takes positive values on $v_1 \ldots, v_m$. Hence so does $\sum_\alpha \chi_\alpha(x)\, (\Lambda_\alpha)_x$. □

EXERCISE 4.17 (Oriented double cover). Let M be a connected manifold. The tangent space at any point $x \in M$ has exactly two orientations. The choice of an orientation on x also induces orientations on "nearby" points. Let $\pi : \hat{M} \to M$ be the map with fibers $\pi^{-1}(x)$ the two possible orientations on x. Show that $\hat{M}$ has a natural structure of an oriented manifold, with π a local diffeomorphism. Show that M is orientable if and only if $\hat{M}$ is disconnected, and the choice of an orientation is equivalent to the choice of a component of $\hat{M}$. One calls $\hat{M}$ the *oriented double cover* of M. What is the oriented double cover for $\mathbb{R}P(2)$? For the Klein bottle?

If M, N are two oriented manifolds, with orientations defined by volume forms Γ_M, Γ_N, the direct product $M \times N$ carries an orientation defined by the volume form $\pi_M^*\Gamma_M \wedge \pi_N^*\Gamma_N$, where π_M, π_N are the projections from $M \times N$ to the two factors.

EXERCISE 4.18. Let $F : N \to M$ be a smooth map between oriented manifolds, and suppose $a \in M$ is a regular value. Then any choice of volume forms Γ_N, Γ_M defines a volume form Γ_S on the level set $S = F^{-1}(a)$. In particular, S is oriented.

Suppose M is an oriented manifold with boundary ∂M. We define an orientation on ∂M as follows: Given $x \in \partial M$, let $v_1 \in T_xM \backslash T_x\partial M$ be an *outward* pointing vector. That is, $v_1(f) \leq 0$ for any function $f \in C^\infty(M)$ with $f \geq 0$ on M. Call a basis $v_2, \ldots, v_m$ of $T_x\partial M$ oriented if $v_1, \ldots, v_m$ is an oriented basis of T_xM.

4.7. Integration on manifolds. Let $U \subset \mathbb{R}^m$ be open. The integral of a compactly supported differential form $\alpha \in \Omega^k(U)$ is defined to be 0 unless $k = m$. For

$$\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \in \Omega^m(U),$$

with f a compactly supported smooth function on U, one defines

$$\int_U \omega = \int \cdots \int f(x^1, \ldots, x^m)\ \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^m$$

as a Riemann integral.

Recall the change of variables formula for the Riemann integral: If $F : V \to U$ is a diffeomorphism, then

$$\int_U \omega = \int \cdots \int (F^* f)(y^1, \ldots, y^m)\ \det(\frac{\partial F^i}{\partial y^j}) \mathrm{d}y^1 \mathrm{d}y^2 \cdots \mathrm{d}y^m$$

Since $\det(\frac{\partial F^i}{\partial y^j}) \mathrm{d}y^1 \mathrm{d}y^2 \cdots \mathrm{d}y^m$ is just the pull-back under F of the form $\mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^m$, this can be written in more compact form,

$$\int_U \omega = \int_{F^{-1}(U)} F^* \omega.$$

This formula shows that integration is independent of the choice of coordinates, and is used to extend integration to manifolds:

THEOREM 4.19. *Let M be an oriented manifold of dimension m. There is a unique linear map $\int_M : \Omega_{\mathrm{comp}}(M) \to \mathbb{R}$ such that for all oriented charts (U, ϕ) of M, and any form ω with compact support in U,*

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \tag{4}$$

PROOF. Choose an atlas (U_α, ϕ_α) and a partition of unity χ_α subordinate to the cover $\{U_\alpha\}$. For any compactly supported form ω on M, define

$$\int_M \omega := \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* (\chi_\alpha \omega).$$

This has the required property (4), since if ω is supported in a chart (U,ϕ), the change of variables formula shows

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\chi_\alpha\omega) = \sum_\alpha \int_{\phi(U)} (\phi^{-1})^*(\chi_\alpha\omega) = \int_{\phi(U)} (\phi^{-1})^*\omega.$$

Conversely, this equation also shows uniqueness of an integration map satisfying (4). $\square$

We will often use the following notation: If M is any manifold, and $S \subset M$ is an oriented embedded submanifold, and $\iota:\, S \to M$ the inclusion map, one defines

$$\int_S :\, \Omega(M) \to \mathbb{R},\ \int_S \omega := \int_S \iota^*\alpha.$$

4.8. Integration over the fiber. Let M, B be manifolds of dimension m, b, and let

$$\pi:\, M \to B$$

be a submersion. We have proved that each fiber $\pi^{-1}(y) \subset M$ of π is a smooth embedded submanifold of dimension $f = m - b$. Moreover, for any $x \in M$ there exists a local chart (U,ϕ) around x such, in the coordinates $x^1, \ldots, x^m$ defined by ϕ, the map π is just projection onto the last b coordinates.

To define the operation "integration over the fibers" we need to assume that each fiber $\pi^{-1}(y)$ is oriented, and that the orientation "depends smoothly on y".

Definition 4.20. A *fiberwise orientation* for the submersion $\pi:\, M \to B$ is an equivalence class of forms $\Gamma_\pi \in \Omega^{m-b}(M)$ such that for each $y \in B$, the pull-back of Γ_π to the fiber $\pi^{-1}(y)$ is a volume form for $\pi^{-1}(y)$. Two such forms Γ_π, Γ'_π are called equivalent if their pull-backs to each fiber $\pi^{-1}(y)$ differ by a positive function on $\pi^{-1}(y)$.

For example, if M, B are oriented by volume forms Γ_M, Γ_B, one obtains a fiberwise orientation by letting Γ_π any $m-b$-form with $\Gamma_M = \Gamma_\pi \wedge \Gamma_B$. The form Γ_π is not uniquely defined by this property, but its pull-back to the fibers is.)

Theorem 4.21. *Let M, B be manifolds of dimension m, b, and $\pi:\, M \to B$ be a submersion with smoothly oriented fibers. There exists a unique linear map*

$$\pi_*:\, \Omega_{\mathrm{comp}}(M) \to \Omega_{\mathrm{comp}}(B)$$

of degree $b - m$ with the following properties:

a) For all $y \in B$ the diagram

$$\begin{array}{ccc} \Omega_{\mathrm{comp}}(M) & \longrightarrow & \Omega_{\mathrm{comp}}(\pi^{-1}(y)) \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \int} \\ \Omega_{\mathrm{comp}}(B) & \longrightarrow & \Omega(\{y\}) = \mathbb{R} \end{array}$$

commutes. Here the horizontal maps are pull-back under the inclusion.

b) For all $\nu \in \Omega_{\mathrm{comp}}(M)$ and all $\alpha \in \Omega(B)$,

$$\pi_*\nu \wedge \alpha = \pi_*(\nu \wedge \pi^*\alpha).$$

The map π_* is called integration over the fiber*,*

Proof. Since π_* has degree $-f = -(m-b)$, it vanishes on forms of degree less then f. On forms of degree f it is determined by property (a). In general, any form on M can be written as a locally finite sum of f-forms on M, wedged with pull-backs of forms on B. (This is easily seen locally, and the global statement follows by choosing a partition of unity.) Hence, uniqueness is clear and it suffices to prove existence. Again, by partition of unity it suffices to prove existence in charts. By the normal form theorem for submersions, we can cover M by charts U, with image a direct product of open sets $\phi(U) = V \times W$, such that π becomes projection to the first $f \leq m$ coordinates. This reduces the theorem to the product situation. □

4.9. Stokes' theorem. Integration can be extended to manifolds M with boundary. Recall that manifolds with boundary are defined similar to manifolds, but taking the half space $x^1 \leq 0$ as the target space for the coordinate charts. The boundary ∂M of M is a manifold of dimension $m-1$. It inherits an orientation from a given orientation on M, as follows: (...) Let $x \in \partial M$, and $v_1 \in T_X M$ an outward-pointing tangent vector at x. A basis $v_2, \ldots, v_n$ a basis of $T_x(\partial M)$ is oriented if and only if $v_1, \ldots, v_n$ is oriented. That is, in local coordinates identifying U with an open subset of $\{x_1 \leq 0\}$ and ∂U with an open subset of the subspace $\{x_1 = 0\}$, the orientation is given by the volume form $\mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^m$.

Theorem 4.22 (Stokes). *If M is an oriented manifold with boundary, and τ is a compactly supported form, then*

$$\int_M d\tau = \int_{\partial M} \tau.$$

Proof. We may assume that the support of τ is contained in some coordinate chart (U, ϕ), where $\phi(U)$ is an open subset of the half space $\{x_1 \leq 0\}$. Write

$$\tau = \sum_{i=1}^m f_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m$$

where each f_i has compact support. Then

$$\mathrm{d}\tau = \sum_{i=1}^m (-1)^{i+1} \frac{\partial f_i}{\partial x_i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

Using Fubini's theorem, we can change the order of integration, and in the ith term integrate over the x^i-variable first. Thus all terms, except the term for $i = 1$, vanish.

We find

$$\begin{aligned}\int_U \mathrm{d}\tau &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\cdots\Big(\int_{-\infty}^{0}\frac{\partial f_1}{\partial x^1}\mathrm{d}x^1\Big)\wedge \mathrm{d}x^2\cdots\wedge \mathrm{d}x^m\\ &= \int_{\partial U} f_1 \mathrm{d}x^2\cdots\wedge \mathrm{d}x^m = \int_{\partial U}\tau.\end{aligned}$$

□

One can view Stokes' theorem as a geometric version of Green's formula from calculus.

Two maps $F_0, F_1 \in C^\infty(S, M)$ are called (smoothly) homotopic if there exists a map $F \in C^\infty([0,1]\times S, M)$ such that $F(0,\cdot) = F_0$ and $F(1,\cdot)$.

Corollary 4.23. *Suppose S is an oriented manifold, and $F_0, F_1 : S \to M$ two smooth maps which are smoothly homotopic. For any* closed *form ω on M,*

$$\int_S F_1^*\omega = \int_S F_0^*\omega.$$

Proof. Using Stokes' theorem,

$$\int_S F_1^*\omega - F_0^*\omega = \int_{[0,1]\times S}\mathrm{d}(F_t^*\omega) = \int_{[0,1]\times S} F_t^*\mathrm{d}\omega = 0.$$

□

The following generalizes Stokes' theorem to submersions.

Theorem 4.24. *If $\pi : M \to B$ is a submersion between oriented manifolds (without boundary),*

$$(-1)^{m-b} d\circ\pi_* = \pi_*\circ d.$$

More generally, if M has a boundary, and $\partial\pi := \pi|_{\partial M}$ is also a submersion, then

$$(-1)^{m-b} d(\pi_*\omega) = \pi_*(d\omega) - (\partial\pi)_*\omega.$$

4.10. Homotopy operators. Suppose $F_0, F_1 \in C^\infty(N, M)$ are called (smoothly) homotopic. Let

$$h := \pi_*\circ F^* : \Omega(M) \to \Omega(N)$$

be the operator of degree -1, defined as a composition of pull-back by F and integration over fibers of $\pi : [0,1]\times N \to N$.

Theorem 4.25. *The map h is a* homotopy operator *between F_0^*, F_1^*. That is,*

$$h\circ d + d\circ h = F_1^* - F_0^* : \Omega(M) \to \Omega(N).$$

Proof. Let ι_0, ι_1 denote the inclusion of the two boundary components $N\times\{0\}$, $N\times\{1\}$. Since the fibers of π are 1-dimensional, we have

$$\pi_*\mathrm{d} + \mathrm{d}\pi_* = (\partial\pi)_* = \iota_1^* - \iota_0^*.$$

Therefore

$$\begin{aligned} h\mathrm{d} + \mathrm{d}h &= \pi_* F^* \mathrm{d} + \mathrm{d}\pi_* F^* \\ &= (\pi_* \mathrm{d} + \mathrm{d}\pi_*) F^* \\ &= (\partial \pi)_* \circ F^* \\ &= (F \circ \iota_1)^* - (F \circ \iota_0)^* \\ &= F_1^* - F_0^*. \end{aligned}$$

□

COROLLARY 4.26 (Poincare Lemma). *Suppose $U \subset \mathbb{R}^m$ is an open ball of radius R around 0, possibly $R = \infty$. If $\omega \in \Omega^k(U)$, with $k > 0$, is closed (i.e. $d\omega = 0$) then ω is exact, i.e. $\omega = d\alpha$ for some α.*

PROOF. Let $F : [0,1] \times U \to U$ be the map $F(t,x) = tx$. Then F_1 is the identity map, and F_0 is the constant map taking everything to the orgin. Thus $F_0^* \omega = 0$ since we assume the degree of ω is positive, and of course $F_1^* \omega = \omega$. Let h be the homotopy operator for F. Then

$$\omega = F_1^* \omega - F_0^* \omega = \mathrm{d}h\omega + h\mathrm{d}\omega = \mathrm{d}h\omega = \mathrm{d}\alpha$$

where $\alpha = h\omega$. □

Note that the homotopy operator provides an explicit primitive for the closed form ω.

EXAMPLES 4.27. Let h be the homotopy operator for $\mathbb{R}^m$, corresponding to the retraction $F(t,x) = tx$. Consider the volume form $\Gamma = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ on $\mathbb{R}^m$. Then

$$F^*\Gamma = \sum_j (-1)^{j+1} t^{m-1} \mathrm{d}t \wedge \sum_j (-1)^j x^j \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^m + \ldots$$

where ... denotes terms not involving $\mathrm{d}t$. Thus

$$h\Gamma = \frac{1}{m} \sum_j (-1)^{j+1} x^j \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^m$$

The right hand sides is proportional to the volume form Ψ on S^{m-1}. More precisely, one has

$$h\Gamma = c\, r^m \Psi$$

where $r = ||x||$ and c is a certain constant.

Next, consider a 1-form $\omega = \sum_i f_i \mathrm{d}x^i$. Then

$$F^*\omega = \sum_i x^i f_i(tx) \mathrm{d}t + \ldots$$

so that

$$h\omega = \sum_i x^i \int_0^1 f_i(tx) \mathrm{d}t.$$

If ω is closed, one checks that $\mathrm{d}h\omega = \omega$.

EXERCISE 4.28. Let $\mathcal{E} = \sum_j x^j \frac{\partial}{\partial x^j}$ be the *Euler vector field* on $\mathbb{R}^m$. Show that if ω is a closed k-form $\omega = \sum_I \omega_I \mathrm{d}x^I$ where the coefficients are polynomials of degree l, we have

$$h\omega = \frac{1}{k+l}\iota(\mathcal{E})\omega.$$

5. De Rham cohomology

5.1. Definition. Let M be a manifold. The exterior differential d on $\Omega(M)$ gives us a *complex*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \longrightarrow 0.$$

called the *de Rham complex*. The subspace of closed forms

$$Z^k(M) = \{\alpha \in \Omega^k(M) | \, \mathrm{d}\alpha = 0\}$$

is called the space of *cocycles*, and the space of exact forms

$$B^k(M) = \mathrm{d}(\Omega^{k-1}(M)) \subset \Omega^k(M)$$

is the space of *coboundaries*. One writes $B(M) = \bigoplus_{k=0}^m B^k(M)$ and $Z(M) = \bigoplus_{k=0}^m Z^k(M)$. Since d squares to zero, every k-coboundary is a k-cocycle. The converse is not true in general, and the obstruction is measured by the *cohomology*

$$H^k(M) = Z^k(M)/B^k(M)$$

We write $Z(M) = \bigoplus_{k=0}^m Z^k(M)$ and $B(M) = \bigoplus_{k=0}^m B^k(M)$ and call

$$H(M) = \bigoplus_{k=0}^m H^k(M)$$

the the *de Rham cohomology* of M. The numbers

$$b_k = \dim H^k(M)$$

are called the *Betti numbers* of the manifold M. (We will see later that $b_k < \infty$ if M is compact.) The polynomial

$$p(t) = \sum_k b_k t^k$$

is called the *Poincare polynomial* of M. Note that $Z(M)$ is a graded algebra under wedge product. Moreover, $B(M)$ is an ideal in $Z(M)$, since $(\mathrm{d}\beta) \wedge \gamma = \mathrm{d}(\beta \wedge \gamma)$ for $\beta \in \Omega(M)$ and $\gamma \in Z(M)$. Thus $H(M)$ becomes a super-commutative graded algebra.

We have seen that on $\mathbb{R}^m$ (and more generally, on star-shaped open subset fo $\mathbb{R}^m$), every closed form of degree $k > 0$ is also exact. Thus

$$H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{otherwise} \end{cases}.$$

EXAMPLES 5.1. (a) Since $B^0(M) = 0$, the zeroth cohomology space is just $H^0(M) = Z^0(M)$. A function $f \in \Omega^0(M)$ is a cocycle if and only is f is constant on connected components of M. Thus $Z^0(M)$ is the space of locally constant functions, and b_0 is simply the number of connected components.

(b) Suppose M is a manifold of dimension m, d vanishes on m-forms since there are no $m+1$-forms. Thus

$$Z^m(M) = \Omega^m(M).$$

When is such a form exact? Suppose M is connected, orientable and compact, and let Γ_M be a volume form on M defining the orientation. Then $\int_M \Gamma > 0$. If $\Gamma = \mathrm{d}\omega$ for some $m-1$-form ω, then we would have $\int_M \Gamma = \int_M \mathrm{d}\omega = 0$ by Stokes' theorem. This contradiction shows that Γ represents a non-trivial cohomology class in $H^m(M)$. We will show later that indeed, if M is compact, orientable and connected $H^m(M)$ is 1-dimensional, and is spanned by the class of Γ. If M is non-compact or non-orientable this is usually false.

(c) Let us try to get a feeling for $H^1(M)$. If $\alpha \in \Omega^1(M)$ is exact, $\alpha = \mathrm{d}f$ then the integral $\int_{S^1} \gamma^*\alpha$ along any closed path $\gamma : S^1 \to M$ vanishes, by Stokes' theorem:

$$\int_{S^1} \gamma^*\alpha = \int_{S^1} \gamma^*\mathrm{d}f = \int_{S^1} \mathrm{d}\gamma^* f = 0.$$

The converse is true as well: If $\alpha \in \Omega^1(M)$ is such that $\int_{S^1} \gamma^*\alpha$ is always 0, then the integral of α along a path $\lambda : [0,1] \to M$ depends only on the end points. One can then define a function f by fixing $x_0 \in M$ and setting $f(x) = \int_{[0,1]} \lambda^*\alpha$ for any path from x_0 to x. It is not hard to see that the function f obtained in this way is smooth and satisfies $\mathrm{d}f = \alpha$. (Indeed, in a coordinate chart around x_0, the function f is just the image under the homotopy operator from Poincare's Lemma.)

Similarly, α is closed if and only if the integral along closed paths does not change under smooth homotopies of the path. We has already seen one direction of this statement. The opposite direction can be seen as follows: Suppose α is not closed. then there exists $x_0 \in M$ with $\mathrm{d}\alpha \neq 0$ at x_0. Choose $X \in \mathfrak{X}(M)$ such that $\iota(X)\mathrm{d}\alpha \neq 0$ near x_0, and a loop γ_0 never tangent to X. Using the flow of ϵX (with $\epsilon > 0$ sufficiently small), we can construct a homotopy $\gamma : [0,1] \times S^1 \to M$ of this loop. Then $\int_{[0,1]\times S^1} \gamma^*\mathrm{d}\alpha \neq 0$, showing by Stokes' theorem that the integral of α along γ_t changes.

EXERCISE 5.2. Show more generally that a k-form ω is closed if and only if for all maps $F : S^k \to M$, the integral of $F^*\omega$ depends only on the homotopy class of F.

5.2. Homotopy invariance. For any $F \in C^\infty(N, M)$, the pull-back map $F^* : \Omega(M) \to \Omega(N)$ is a homomorphism of graded algebras commuting with d. Hence it defines an algebra homomorphism $F^* : H(M) \to H(N)$.

Theorem 5.3 (Homotopy invariance). *If $F_0, F_1 \in C^\infty(N, M)$ are smoothly homotopic, then the induced maps $F_0^*, F_1^* : H(M) \to H(N)$ are equal.*

Proof. We have to show that the map $F_1^* - F_0^* : \Omega(M) \to \Omega(N)$ induces the zero map in cohomology. Thus, we have to show that if $\beta \in Z(M)$ then $F_1^*\beta - F_0^*\beta \in B(N)$. Let $h : \Omega(N) \to \Omega(M)$ be the homotopy operator for a smooth homotopy between F_0, F_1:

$$F_1^* - F_0^* = h \circ \mathrm{d} + \mathrm{d} \circ h : \Omega(M) \to \Omega(N).$$

Apply this equation to β, using $\mathrm{d}\beta = 0$:

$$F_1^*\beta - F_0^*\beta = \mathrm{d}(h\beta).$$

□

Two maps $F : N \to M$ and $G : M \to N$ are called *homotopy inverses* if $F \circ G$ is homotopic to Id_M and $G \circ F$ homotopic to Id_N. In this case, the theorem shows:

Corollary 5.4. *Suppose $F : N \to M$ and $G : M \to N$ are homotopy inverses. Then $F^* : H(M) \to H(N)$ is an isomorphism with inverse $G^* : H(N) \to H(M)$.*

Proof. According to the theorem, the maps $G^* \circ F^* : H(M) \to H(M)$ and $F^* \circ G^* : H(N) \to H(N)$ are the identity maps. □

Definition 5.5. Let N be a submanifold of a manifold M. A smooth map $F : [0,1] \times M \to M$ such that $F(0, \cdot) = \mathrm{Id}_M$ and $F(1, x) \in N$ for all $x \in M$ is called a *deformation retraction* from M onto N. It is called a *strong deformation retraction* if $F(t, x) = x$ for all $t \in [0, 1]$ and $x \in N$.

As a special case, one has:

Proposition 5.6. *If M admits a strong deformation retraction onto an embedded submanifold N, then the inclusion map $\iota : N \to M$ gives an isomorphism $\iota^* : H(M) \to H(N)$.*

Proof. Let $F : [0,1] \times M \to M$ be a strong deformation retraction. Let $\pi : M \to N$ be the map thus obtained, i.e. $\iota \circ \pi = F(1, \cdot) : M \to N$. Then F gives a homotopy between Id_M and $\iota \circ \pi$. But $\pi \circ \iota = \mathrm{Id}_N$. Thus ι and π are homotopy inverses; in particular ι^* is an isomorphism in cohomology. □

This Proposition once again explains Poincare's Lemma: Since $\mathbb{R}^m$ admits a strong deformation retraction onto a point, its cohomology is trivial.

6. Mayer-Vietoris

6.1. Exact sequences. A (finite or infinite) sequence of vector spaces and maps

$$\cdots \longrightarrow E \longrightarrow E' \longrightarrow E'' \longrightarrow \cdots$$

is called *exact at* E' if the kernel of the map $E' \to E''$ is equal to the image of the map $E \to E'$. The sequence is called exact if it is exact everywhere. An exact sequence of the form

$$0 \longrightarrow F \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 0$$

is called *short exact*. This has the following interpretation: Exactness at F means that the map $\iota: F \to E$ is injective. That is, we can think of F as a subspace of E. Exactness at G means that the map $\pi: E \to G$ is surjective. Thus we may think of G as a quotient space $G = E/\ker(\pi)$. Exactness at E tells us that $\ker(\pi) = \mathrm{im}(\iota) \cong F$. Putting all this together, we conclude that F is a subspace of E and $G = E/F$.

Exercise 6.1. Let $A: F \to E$ be any linear map. Show that the sequence

$$0 \longrightarrow \ker(A) \longrightarrow F \xrightarrow{A} E \longrightarrow E/\,\mathrm{im}(A) \longrightarrow 0$$

is exact.

Suppose the sequence

$$0 \longrightarrow E \xrightarrow{\iota} E' \xrightarrow{k} E'' \longrightarrow \cdots$$

is exact. Then E can be thought of as a subspace of E', and the map $k: E' \to E''$ vanishes exactly on $E \subset E'$. That is, k descends to an injective map $E'/E \to E''$. Thus we obtain a new exact sequence

$$0 \longrightarrow\longrightarrow E'/E \xrightarrow{k} E'' \longrightarrow \cdots$$

Exercise 6.2. Show that if the sequence $0 \to E^0 \to \cdots \to E^k \to 0$ is exact, then the alternating sum of dimensions vanishes:

$$\sum_i (-1)^i \dim E^i = 0.$$

6.2. Differential complexes. The de Rham complex $(\Omega(M), \mathrm{d})$ is an example of a *differential complex*. Since we will encounter many more examples of differential complexes later on, it is useful to treat them in generality.

Let $E = \bigoplus_{k\in\mathbb{Z}} E^k$ be a graded vector space, and $\mathrm{d}: E \to E$ a linear map of degree 1. That is, d is a collection of maps

$$\cdots \longrightarrow E^k \xrightarrow{d} E^{k+1} \xrightarrow{d} E^{k+2} \longrightarrow \cdots \tag{5}$$

One calls (E, d) a *differential complex* if d squares to 0. The kernel of the map $\mathrm{d}: E^k \to E^{k+1}$ is called the space of k-*cocycles*, and is denoted $Z^k(E)$. The image of the map $\mathrm{d}: E^{k-1} \to E^k$ is called the space of k-*coboundaries* and is denoted $B^k(E)$. Since $\mathrm{d} \circ \mathrm{d} = 0$, $B^k(E)$ is a subspace of $Z^k(E)$. The quotient space

$$H^k(E) = Z^k(E)/B^k(E)$$

is called the kth cohomology of the complex (E, d). We denote $H(E) = \bigoplus_{k\in\mathbb{Z}} H^k(E)$. Note that $H(E) = 0$ if and only if the sequence (5) is exact.

(In the special case $E = \Omega(M)$, we will continue to write $B(M), Z(M), H(M)$ rather than $B(\Omega(M)), Z(\Omega(M), H(\Omega(M))$.

Definition 6.3. Let (E, d) and (E', d) be two differential complexes. A linear map $\Phi : E \to E'$ of degree 0 is called a *homomorphism of differential complexes* or *cochain map*, if $\Phi \circ \mathrm{d} = \mathrm{d} \circ \Phi$.

A cochain map $\Phi : E \to E'$ takes cocycles to cocycles, hence it gives a map in cohomology, which we denote by the same letter:

$$\Phi : H(E) \to H(E').$$

For example, if $F : M \to N$ is a smooth map of manifolds, it defines a cochain map $\Phi = F^* : \Omega(N) \to \Omega(M)$, and therefore $H(N) \to H(M)$.

Definition 6.4. A *homotopy operator* (or "*chain homotopy*") between two cochain maps $\Phi_0, \Phi_1 : E \to E'$ is a linear map $h : E' \to E$ of degree -1, with property

$$h \circ \mathrm{d} + \mathrm{d} \circ h = \Phi_1 - \Phi_0.$$

If such a map h exists, Φ_0, Φ_1 are called *chain homotopic.* Two cochain maps $\Phi : E \to E'$ and $\Psi : E' \to E$ are called *homotopy inverse* if $\Phi \circ \Psi : F \to F$ and $\Psi \circ \Phi : E \to E$ are both homotopic to the identity, if such maps exist, E, E' are called *homotopy equivalent.*

For example, if $F_0, F_1 : N \to M$ are smoothly homotopic, we had constructed a homotopy operator between $\Phi_1 = F_1^*$ and $\Phi_0 = F_0^*$. As in this special case, we see:

Proposition 6.5. *Chain homotopic cochain maps induce equal maps in cohomology. If two differential complexes E, E' are homotopy equivalent, their cohomologies are equal.*

Suppose now that E, F, G are differential complexes, and let

$$0 \longrightarrow E \xrightarrow{j} F \xrightarrow{k} G \longrightarrow 0$$

be a short exact sequence where all maps are homomorphisms of differential complexes. We will construct a map $\delta : H(G) \to H(E)$ of degree $+1$ called the *connecting homomorphism.* This is done as follows: Let $[\gamma] \in H^k(G)$ be a given cohomology class, represented by $\gamma \in Z^k(G)$. Since $k : F \to G$ is surjective, we can choose, $\beta \in F^k$ with $k(\beta) = \gamma$. Then

$$k(\mathrm{d}\beta) = \mathrm{d}(k(\beta)) = \mathrm{d}\gamma = 0.$$

Thus, $\mathrm{d}\beta \in \ker(k) = \mathrm{im}(j)$. Since j is injective, there is a unique $\alpha \in E^{k+1}$ with $j(\alpha) = \mathrm{d}\beta$. Then $j(\mathrm{d}\alpha) = \mathrm{d}j(\alpha) = \mathrm{d}\mathrm{d}\beta = 0$, so $\mathrm{d}\alpha = 0$. Set

$$\delta([\gamma]) = [\alpha].$$

One has to check that this definition does not depend on the choices of γ and β. For example, if $\beta' \in F^k$ is another choice with $k(\beta') = \gamma$, then $k(\beta' - \beta) = 0$, so $\beta' - \beta \in \ker(k) = \mathrm{im}(j)$. Since j is injective, there is a unique element $\phi \in E^k$ with $j(\phi) = \beta' - \beta$. Thus if $\alpha' \in E^{k+1}$ is the unique element such that $j(\alpha') = \mathrm{d}\beta'$, we have $\alpha' - \alpha = \mathrm{d}\phi$, so $[\alpha'] = [\alpha]$. Similarly, one checks that the definition does not depend on the choice of representative γ for $[\gamma]$.

THEOREM 6.6. *Let*

$$0 \longrightarrow E \xrightarrow{j} F \xrightarrow{k} G \longrightarrow 0$$

be a short exact sequence of homomorphisms of differential complexes with connecting homomorphism δ. Then there is a long exact sequence in cohomology,

$$\cdots \longrightarrow H^k(E) \xrightarrow{j} H^k(F) \xrightarrow{k} H^k(G) \xrightarrow{\delta} H^{k+1}(E) \longrightarrow \cdots$$

where the connecting homomorphism $\delta : H^k(G) \to H^{k+1}(E)$ is defined as follows: For $\gamma \in Z^k(G)$, $\delta([\gamma]) = [\alpha]$ where $\alpha \in Z^{k+1}(E)$ is an element with $j(\alpha) = d\beta$ for $k(\beta) = \gamma$.

PROOF. Let us check exactness at $H^k(G)$. We have to check two inclusions, (i)im$(k) \subset$ ker(δ) and (ii) ker$(\delta) \subset$ im(k).

(i) Suppose $\beta \in Z^k(F)$. We have to show that $\delta[k(\beta)] = 0$. By construction of the connecting homomorphism, $\delta[k(\beta)] = [\alpha]$ where $j(\alpha) = \mathrm{d}\beta = 0$. Thus $\alpha = 0$.

(ii) Suppose $\gamma \in Z^k(G)$ with $\delta[\gamma] = 0$. We will show that there exists a *closed* element $\beta \in Z^k(F)$ such that $\gamma = k(\beta)$. Start by choosing *any* $\beta \in F^k$ with $k(\beta) = \gamma$, and let $\alpha \in E^{k+1}$ be the unique element with $j(\alpha) = \mathrm{d}\beta$. By definition of δ, $0 = \delta[\gamma] = [\alpha]$, thus $\alpha = \mathrm{d}\phi$ for some $\phi \in E^k$. Define $\beta' = \beta - j(\phi)$. Then β' is closed:

$$\mathrm{d}\beta' = \mathrm{d}\beta - j(\mathrm{d}\phi) = \mathrm{d}\beta - j(\alpha) = 0,$$

and $k(\beta') = k(\beta) = \gamma$ since $k \circ j = 0$. Exactness at $H^k(F)$ and $H^k(E)$ is checked similarly. □

THEOREM 6.7 (Mayer-Vietoris). *Suppose $M = U \cup V$ where U, V are open, and let χ_U, χ_V be a partition of unity subordinate to the cover of U, V. The sequence*

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{j} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{k} \Omega^p(U \cap V) \longrightarrow 0$$

where $j(\alpha) = (\alpha|_U, \alpha|_V)$ and $k(\beta, \beta') = \beta|_{U\cap V} - \beta'|_{U\cap V}$ is exact. It induces a long exact sequence

$$0 \longrightarrow \cdots \longrightarrow H^p(U\cup V) \longrightarrow H^p(U)\oplus H^p(V) \longrightarrow H^p(U\cap V) \xrightarrow{\delta} H^{p+1}(U\cup V) \longrightarrow \cdots \longrightarrow 0$$

The connecting homomorphism takes the cohomology class of $\gamma \in Z^p(U \cap V)$ to the class of $\delta(\gamma) \in Z^{p+1}(U \cup V)$, where

$$\delta(\gamma)|_U = d(\chi_V \gamma), \quad \delta(\gamma)|_V = -d(\chi_U \gamma).$$

The formula for $\delta(\gamma)$ makes sense, since on the overlap $U \cap V$,

$$\mathrm{d}(\chi_V\gamma) - (-\mathrm{d}(\chi_U\gamma)) = \mathrm{d}((\chi_U + \chi_V)\gamma) = \mathrm{d}\gamma = 0.$$

PROOF. Exactness at $\Omega^p(M)$ is obvious since the map $\Omega^p(M) \to \Omega^p(U) \oplus \Omega^p(V)$ is clearly injective. The kernel of the map $\Omega^p(U) \oplus \Omega^p(V) \longrightarrow \Omega^p(U \cap V)$ consists of forms β, β' with $\beta|_{U\cap V} = \beta'|_{U\cap V}$. These are exactly the forms which patch together to a global form α on M, with $\beta = \alpha|_U$ and $\beta' = \alpha|_V$. This shows exactness at $\Omega^p(U) \oplus \Omega^p(V)$. It remains to check surjectivity of the last map $\psi : \Omega^p(U) \oplus \Omega^p(V) \longrightarrow \Omega^p(U \cap V)$. Let $\alpha \in \Omega^p(U \cap V)$. Choose a partition of unity χ_U, χ_V subordinate to the cover U, V. Then $\chi_V \alpha$ is zero in a neighborhood of $U \backslash U \cap V$, hence it extends by 0 to a form β on U. Similarly $-\chi_U \alpha$ is zero in a neighborhood of $V \backslash U \cap V$, hence it extends by 0 to a form β' on V.

On $U \cap V$ we have $\beta|_{U \cap V} - \beta'|_{U \cap V} = \alpha$. This shows that k is surjective. Any short exact sequence of chain complexes induces a long exact sequence in cohomology. □

As a first application of the Mayer-Vietoris sequence, we can now compute the cohomology of the sphere.

Example 6.8. Let $M = S^m$, and U, V the covering by the complements of the south pole and north pole, respectively. The intersection $U \cap V$ smoothly retracts onto the equatorial S^{m-1}. Since $H^k(U) = H^k(V) = 0$ for $k > 0$, Mayer-Vietoris gives isomorphisms

$$H^k(S^n) \cong H^{k-1}(U \cap V) = H^{k-1}(S^{n-1})$$

for $k > 1$. By induction, we get $H^k(S^n) = H^1(S^{n-k+1})$. In low degrees, if $n > 1$, Mayer-Vietoris is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(S^n) \longrightarrow 0.$$

Here the map $\mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}$ is easily seen to be surjective, so $\mathbb{R} \longrightarrow H^1(S^n)$ must be the zero map, so $H^1(S^n) = 0$. For $n = 1$ we have $H^1(S^1) = 0$. We conclude,

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, n, \\ 0 & \text{otherwise} \end{cases} .$$

Thus $p(t) = 1 + t^n$ is the Poincare polynomial.

7. Compactly supported cohomology

If M is non-compact, one can also study the *compactly supported* de Rham cohomology: Working with the complex of compactly supported forms

$$0 \longrightarrow \Omega^0_{\mathrm{comp}}(M) \xrightarrow{d} \Omega^1_{\mathrm{comp}}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_{\mathrm{comp}}(M) \longrightarrow 0$$

one defines $H^p_{\mathrm{comp}}(M)$ as the cohomology of this complex,

$$H^p_{\mathrm{comp}}(M) = H^p(\Omega^m_{\mathrm{comp}}(M)).$$

If M is compact, $H_{\mathrm{comp}}(M)$ agrees with the usual cohomology, but for non-compact manifolds it is quite different. For instance, if M has no compact component,

$$H^0_{\mathrm{comp}}(M) = 0.$$

This follows since $Z^0_{\mathrm{comp}}(M)$ consists of locally constant functions *of compact support*, and there are no such if M has no compact component. It is also different in higher degree: For instance,

$$H^1_{\mathrm{comp}}(\mathbb{R}) = \mathbb{R} \tag{6}$$

in contrast to $H^1(\mathbb{R}) = 0$. To see (6), let $\alpha \in \Omega^1_{\mathrm{comp}}(\mathbb{R})$. Write $\alpha = f\,\mathrm{d}t$, where t is the coordinate on $\mathbb{R}$, and $f \in C^\infty_{\mathrm{comp}}(\mathbb{R})$. Suppose $\alpha = \mathrm{d}F$. Then, by the fundamental theorem of calculus, $F(t) = \int_{-\infty}^t f(s)\mathrm{d}s + c$, where c is a constant. Let $T > 0$ so that $\mathrm{supp}(f) \subset [-T, T]$. Then $F(t)$ is equal to $F(\infty) = c + \int \alpha$ for $t > T$ and equal to $F(-\infty) = c$ for $t < -T$. This shows that $\alpha = \mathrm{d}F$ for a *compactly supported* function F, if and only if $\int_{\mathbb{R}} \alpha = 0$. If we take f to be a "bump function", i.e. a compactly supported function of integral 1, it follows that the class of $\alpha_0 = f\mathrm{d}t \in \Omega^1_{\mathrm{comp}}(\mathbb{R})$ generates $H^1_{\mathrm{comp}}(\mathbb{R})$. Indeed, $[\alpha_0] \in H^1_{\mathrm{comp}}(\mathbb{R})$ is non-zero, and for any $\alpha \in \Omega^1_{\mathrm{comp}}(\mathbb{R})$, we have $\alpha - c\alpha_0 \in B^1_{\mathrm{comp}}(\mathbb{R})$ (thus $[\alpha] = c[\alpha_0]$) since its integral is zero.

This example shows that in general, the map

$$H^k_{\mathrm{comp}}(M) \to H^k(M).$$

induced by the inclusion $Z^k_{\mathrm{comp}}(M) \to Z^k(M)$ is neither injective nor surjective.

THEOREM 7.1. *Let M be a compact manifold, and $x \in M$. Then*

$$H^k_{\mathrm{comp}}(M\backslash\{x\}) = H^k(M)$$

for all $k > 0$.

PROOF. Let $\omega \in \Omega^k(M)$ be closed. On any contractible open neighborhood U of x, $\omega|_U = \mathrm{d}\beta$ for some form $\beta \in \Omega(U)$. Choose a function $\chi \in C^\infty(M)$ supported on U with $\chi = 1$ near x. Then $\omega - \mathrm{d}(\chi\beta)$ is cohomologous to ω and vanishes near x. This shows that the natural map $H^k_{\mathrm{comp}}(M\backslash\{x\}) \to H^k(M)$ is surjective. Let us now show that it is injective. Suppose $\omega \in \Omega^k_{\mathrm{comp}}(M\backslash\{x\})$ is closed as a form on M, that is $\omega = \mathrm{d}\beta$ where $\beta \in \Omega(M)$. We have to show that β can be chosen to be 0 near x.

Let U be a contractible open neighborhood of x such that ω vanishes near x. Then $\mathrm{d}\beta = 0$ on U, i.e. β is closed on U. If $k > 1$, we can argue as before: Write $\beta|_U = \mathrm{d}\gamma$ and replace β by $\beta - \mathrm{d}(\chi\gamma)$. This shows that ω is also the differential of a compactly supported form. If $k = 1$, β has degree 0 so this argument doesn't work. But in this case, $f = \beta$ is just a function which equals a constant $a = f(x)$ near x. We may replace f by $f - a$ to produce a compactly supported function with differential ω. □

THEOREM 7.2.

$$H^k_{\mathrm{comp}}(\mathbb{R}^m) = \begin{cases} \mathbb{R} & \text{for } k = m, \\ 0 & \text{otherwise} \end{cases} .$$

PROOF. This follows from the cohomology of S^m, since $\mathbb{R}^m$ is diffeomorphic to S^m minus a point. □

Let us discuss some of the properties of H_{comp}. If $F \in C^\infty(M, N)$ is a *proper* map, pull-back F^* induces a chain map $\Omega_{\mathrm{comp}}(N) \to \Omega_{\mathrm{comp}}(M)$ hence a map $H_{\mathrm{comp}}(N) \to H_{\mathrm{comp}}(M)$. (A map is called proper if the pre-image of any compact set is compact.)

One proves as before that H_{comp} is homotopy invariant, but only under *proper* homotopies. This is why, for example, the compactly supported cohomology of $\mathbb{R}^m$ is *not* the cohomology of a point.

Similarly, if $U \subset M$ is an open subset, one has a natural restriction map

$$\Omega(M) \to \Omega(U),$$

but this map does not take $\Omega_{\mathrm{comp}}(M)$ into $\Omega_{\mathrm{comp}}(U)$ (unless U is a connected component of M, the inclusion map is not proper.) On the other hand, one has a natural "extension by 0" map

$$\Omega_{\mathrm{comp}}(U) \to \Omega_{\mathrm{comp}}(M),$$

but there is no such map from $\Omega(U)$ to $\Omega(M)$. As a consequence, the Mayer-Vietoris sequence for $\Omega_{\mathrm{comp}}(\cdot)$ looks different from that for $\Omega(\cdot)$.

THEOREM 7.3 (Mayer-Vietoris for cohomology with compact support). *Suppose $M = U \cup V$ where U, V are open, and let χ_U, χ_V be a partition of unity subordinate to the cover $\{U, V\}$. The sequence*

$$0 \longrightarrow \Omega^p_{\mathrm{comp}}(U \cap V) \xrightarrow{j} \Omega^p_{\mathrm{comp}}(U) \oplus \Omega^p_{\mathrm{comp}}(V) \xrightarrow{k} \Omega^p_{\mathrm{comp}}(U \cup V) \longrightarrow 0,$$

where $j(\alpha) = (\alpha, -\alpha)$ and $k(\beta, \beta') = \beta + \beta'$, is exact for all p. Hence there is a long exact sequence in compactly supported cohomology,

$$\cdots \to H^p_{\mathrm{comp}}(U \cap V) \to H^p_{\mathrm{comp}}(U) \oplus H^p_{\mathrm{comp}}(V) \to H^p_{\mathrm{comp}}(U \cup V) \xrightarrow{\delta} H^{p+1}_{\mathrm{comp}}(U \cap V) \to \cdots .$$

The connecting homomorphism $\delta : H^p_{\mathrm{comp}}(U \cup V) \to H^{p+1}_{\mathrm{comp}}(U \cap V)$ takes the class of $\gamma \in Z^p_{\mathrm{comp}}(U \cup V)$ to the class of

$$\delta(\gamma) := d\chi_U \wedge \gamma = -d\chi_V \wedge \gamma.$$

Note that the formula for $\delta(\gamma)$ is well-defined since $\mathrm{d}\chi_U = -\mathrm{d}\chi_V \in \Omega^1_{\mathrm{comp}}(U \cap V)$.

PROOF. This is very similar to the proof of Mayer-Vietoris for $\Omega(\cdot)$, and is left as an exercise. □

Again, the direct sum $H_{\mathrm{comp}}(M)$ is an algebra under wedge product. Since the wedge product of a compactly supported form with any form is compactly supported, we see that $H_{\mathrm{comp}}(M)$ is a module for the algebra $H(M)$.

8. Finite-dimensionality of de Rham cohomology

Using the Mayer-Vietoris sequence, we will now show that $H(M)$ is for any compact manifold M, or more generally for all manifolds of so-called "finite type".

DEFINITION 8.1. A cover $\{U_\alpha\}_{\alpha \in A}$ is called a *good cover* if all finite non-empty intersections of the sets U_α are diffeomorphic to $\mathbb{R}^m$. A manifold admitting a finite good cover is called of *finite type*.

THEOREM 8.2. *Every given cover of a manifold M has a refinement which is a good cover.*

Sketch of proof. We won't give the full proof here, which requires some elements from Riemannian geometry. The idea is as follows. Choose a Riemannian metric g on M. That is, g is a symmetric $C^\infty(M)$-bilinear form $\mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, with the property that $g(X,X)_x > 0$ if $X_x \neq 0$. Using g one can define the length of curves $\gamma : [a,b] \to M$ as $\mathrm{length}(\gamma) = \int_a^b g(\dot\gamma, \dot\gamma)^{1/2}\, \mathrm{d}t$. An open set $U \subset M$ is called *geodesically convex*, that is any two points in U are joined by a unique (up to re-parametrization) curve $\gamma : I \to U$ of shortest length.

Geodesically convex open sets are diffeomorphic to $\mathbb{R}^m$, and the intersection of two geodesically convex sets is again geodesically convex. Thus any cover consisting of geodesically convex open sets is a good cover. It can be shown that for any point $x \in M$, there exists $\epsilon > 0$ such that the set U of points that can be joined by a curve of length $< \epsilon$ is a geodesically convex open neighborhood of x. This shows that any cover can be refined to a good cover. □

In particular, every compact manifold is of finite type.

Theorem 8.3. *For any manifold of finite type, the de Rham cohomology $H(M)$ and the compactly supported de Rham cohomology $H_{\mathrm{comp}}(M)$ are finite dimensional.*

Proof. Suppose U, V are open subsets of M, and suppose $H(U),\ H(V),\ H(U \cap V)$ are all finite dimensional. Exactness of the Mayer-Vietoris sequence

$$\longrightarrow H^{k-1}(U \cap V) \xrightarrow{\alpha} H^k(U \cup V) \xrightarrow{\beta} H^k(U) \oplus H^k(V) \longrightarrow \cdots$$

shows

$$\begin{aligned} \dim(H^k(U \cup V)) &= \dim(\ker\beta) + \dim(\mathrm{im}\,\beta) \\ &= \dim(\mathrm{im}\,\alpha) + \dim(\mathrm{im}\,\beta) \\ &\le \dim H^{k-1}(U \cap V) + \dim H^k(U) + \dim H^k(V) < \infty. \end{aligned}$$

Now let $U_1, \ldots, U_N$ be a finite good cover of M. For each $l \le N$, the open subsets $U_1 \cup \ldots \cup U_l$ and $(U_1 \cup \ldots \cup U_l) \cap U_{l+1} = \bigcup_{j \le l}(U_j \cap U_{l+1})$ have a finite good cover by l open sets. Hence, by induction on l they all have finite dimensional de Rham cohomology, and by the Mayer-Vietoris argument it follows that $U_1 \cup \ldots \cup U_l \cup U_{l+1}$ has finite dimensional de Rham cohomology. A similar argument applies to cohomology with compact supports. □

Note that the theorem is not true, in general, for manifolds of infinite type: For instance, any manifold with an infinite number of components certainly has infinite-dimensional H^0.

9. Poincare duality

Let M be an oriented manifold. Recall that we have defined a linear map

$$\int_M : \Omega_{\mathrm{comp}}(M) \to \mathbb{R}$$

equal to zero on forms of degree $k < \dim M$. Since the wedge product of a compactly supported form and any form is compactly supported, we can use this to define a bilinear form

$$\Omega(M) \times \Omega_{\mathrm{comp}}(M) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$

Since $\int_M \circ \mathrm{d} = 0$ (Stokes' theorem), integration descends to a linear map

$$\int_M : H_{\mathrm{comp}}(M) \to \mathbb{R}.$$

Thus we have a bilinear pairing

$$H(M) \times H_{\mathrm{comp}}(M) \to \mathbb{R}, \ ([\alpha, \beta]) \mapsto \int_M \alpha \wedge \beta.$$

Let $P : H(M) \to H_{\mathrm{comp}}(M)^*$ be the map defined by

$$P([\alpha])([\beta]) = \int_M \alpha \wedge \beta.$$

Note that P takes $H^p(M)$ to $H^{m-p}_{\mathrm{comp}}(M)$. The main theorem of this section is:

THEOREM 9.1 (Poincare duality). *The map $P : H^p(M) \to H^{m-p}_{\mathrm{comp}}(M)^*$ is an isomorphism for all p.*

The proof is based on the following

LEMMA 9.2. *Let $M = U \cup V$ where U, V are open subsets of M. Then there is a commutative diagram*

$$\begin{array}{ccccccccc}
\longrightarrow H^p(U \cup V) & \longrightarrow & H^p(U) \oplus H^p(V) & \longrightarrow & H^p(U \cap V) & \longrightarrow & H^{p+1}(U \cup V) \longrightarrow \\
\downarrow {\scriptstyle (-1)^{p-1} D} & & \downarrow {\scriptstyle D} & & \downarrow {\scriptstyle D} & & \downarrow {\scriptstyle (-1)^p D} \\
\longrightarrow H^{m-p}_{\mathrm{comp}}(U \cup V)^* & \longrightarrow & H^{m-p}_{\mathrm{comp}}(U)^* \oplus H^{m-p}_{\mathrm{comp}}(V)^* & \longrightarrow & H^{m-p}_{\mathrm{comp}}(U \cap V)^* & \longrightarrow & H^{m-p-1}_{\mathrm{comp}}(U \cup V)^* \longrightarrow
\end{array}$$

Here the upper horizontal map is the Mayer-Vietoris sequence in cohomology, and the lower horizontal map is the dual of the Mayer-Vietoris sequence in compactly supported cohomology.

PROOF. Let $\alpha \in Z(U \cup V)$ and $(\beta, \beta') \in Z_{\mathrm{comp}}(U) \oplus Z_{\mathrm{comp}}(V)$. We have

$$\int_U \alpha|_U \wedge \beta + \int_V \alpha|_V \wedge \beta' = \int_{U \cup V} \alpha \wedge (\beta + \beta')$$

which shows commutativity of the first square. Commutativity of the second square is obtained similarly. To prove commutativity of the third square, let χ_U, χ_V be a partition of unity for U, V. Let $\gamma \in Z^p(U \cap V)$ and $\alpha \in Z^{m-p-1}_{\mathrm{comp}}(U \cup V)$. We have to show

$$\int_{U \cup V} \delta(\gamma) \wedge \alpha = (-1)^p \int_{U \cap V} \gamma \wedge \delta(\alpha).$$

Recall that

$$\delta(\gamma)|_U = \mathrm{d}(\chi_V \gamma), \quad \delta(\gamma)|_V = -\mathrm{d}(\chi_U \gamma).$$

and that

$$\delta(\alpha) = \mathrm{d}(\chi_U \alpha) = \mathrm{d}\chi_U \wedge \alpha.$$

We have

$$(\delta(\gamma) \wedge \alpha)|_U = \mathrm{d}(\chi_V) \wedge \gamma|_U \wedge \alpha|_U = (-1)^p \gamma|_U \wedge \mathrm{d}\chi_V \wedge \alpha|_U.$$

Since $\mathrm{d}\chi_V \wedge \alpha|_U$ is compactly supported in $U \cap V$, we see that $(\delta(\gamma) \wedge \alpha)|_U$ is supported in $U \cap V$. Switching the roles of U, V, we find that $\delta(\gamma) \wedge \alpha$ is supported in $U \cap V$, and compute

$$\int_{U \cup V} \delta(\gamma) \wedge \alpha = (-1)^p \int_{U \cap V} \gamma \wedge \delta(\alpha),$$

as desired. □

Proof of Poincare duality. We give the argument for finite type manifolds; the general case is covered in Greub-Halperin-Vanstone. As in the proof of finite dimensionality of $H(M)$, the proof is based on induction on the number of elements in a good cover $(U_i)_{i=1}^N$. For $l \leq N$, let $U = \bigcup_{i \leq l} U_i$, $V = U_{l+1}$. Then the induction hypothesis applies to $U, V, U \cap V$. Hence the corresponding Poincare duality maps are all isomorphisms. The following algebraic fact implies that $H(U \cup V) \to H_{\mathrm{comp}}(U \cup V)^*$ is then an isomorphism as well. □

Lemma 9.3 (Five-Lemma). *Let*

$$\begin{array}{ccccccccc}
E_1 & \xrightarrow[j_1]{} & E_2 & \xrightarrow[j_2]{} & E_3 & \xrightarrow[j_3]{} & E_4 & \xrightarrow[j_4]{} & E_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
F_1 & \xrightarrow[k_1]{} & F_2 & \xrightarrow[k_2]{} & F_3 & \xrightarrow[k_3]{} & F_4 & \xrightarrow[k_4]{} & F_5
\end{array}$$

be a commutative diagram of vector spaces and linear maps. Suppose the rows are exact, and that $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ *are isomorphisms. Then* α_3 *is also an isomorphism.*

Proof. We will only show injectivity of the map α_3. The proof of surjectivity is very similar and is left as an exercise. Suppose $x_3 \in E_3$ is in the kernel of α_3. We have to show $x_3 = 0$. Since

$$\alpha_4(j_3(x_3)) = k_3(\alpha_3(x_3)) = 0,$$

and α_4 is injective, we have $j_3(x_3) = 0$. Thus $x_3 \in \ker(j_3) = \mathrm{im}(j_2)$. Thus we can write $x_3 = j_2(x_2)$. We have

$$k_2(\alpha_2(x_2)) = \alpha_3(j_2(x_2)) = \alpha_3(x_3) = 0.$$

Thus $\alpha_2(x_2) \in \ker(k_2) = \mathrm{im}(k_1)$. Thus we can write $\alpha_2(x_2) = k_1(y_1)$. Since α_1 is surjective, we can write $y_1 = \alpha_1(x_1)$. Then

$$\alpha_2(j_1(x_1)) = k_1(\alpha_1(x_1)) = k_1(y_1) = \alpha_2(x_2).$$

Since α_2 is injective, this shows $j_1(x_1) = x_2$. Thus $x_2 \in \mathrm{im}(j_1) = \ker(j_2)$. It follows that $x_3 = j_2(x_2) = 0$. $\square$

Poincare duality has many applications. As a first consequence, we can finally get a handle on the top degree cohomology.

Corollary 9.4. *For every connected oriented manifold M, one has $H^m_{\mathrm{comp}}(M) = \mathbb{R}$, and*

$$H^m(M) = \begin{cases} \mathbb{R} & \text{if } M \text{ is compact,} \\ 0 & \text{if } M \text{ is non-compact.} \end{cases}$$

If M is connected but non-orientable, one has $H^m(M) = H^m_{\mathrm{comp}}(M) = 0$.

Proof. For the case M oriented, all the facts are immediate by Poincare duality, and using that $H^0_{\mathrm{comp}}(M) = 0$ if M is non-compact, and equal to $\mathbb{R}$ if M is compact. Note that if M is compact, the non-trivial generator of $H^m(M)$ is represented by a volume form. Suppose now that M is non-orientable. Let $\tilde{M}$ be the oriented double cover. Let $\sigma : \tilde{M} \to \tilde{M}$ be the orientation-reversing diffeomorphism inducing the identity on M. The de-Rham complex $\Omega(\tilde{M}) = \Omega(\tilde{M})_+ \oplus \Omega(\tilde{M})_-$ splits into the direct sum of σ-invariant and anti-invariant forms. Both summands are invariant under d, and $\Omega(\tilde{M})_+$ is identified with $\Omega(M)$. Similarly $H(\tilde{M}) = H(\tilde{M})_+ \oplus H(\tilde{M})_-$, where $H(\tilde{M})_+ \cong H(M)$. If M is non-compact, we conclude $H^m(M) = 0$ since $H^m(\tilde{M}) = 0$. If M is compact, then so is $H(\tilde{M})$. Since the volume form is anti-invariant, its class is in $H^m(\tilde{M})_-$. Thus $H^m(M) = H^m(\tilde{M})_+ = 0$. $\square$

Suppose M is a manifold, and $S \subset M$ a compact, oriented, embedded submanifold of dimension k. Integration over S defines a linear map $\int_S : H^k(M) \to \mathbb{R}$, in other words it is an element of $H^k(M)^*$. Let $[\tau] \in H^{m-p}_{\mathrm{comp}}(M)$ be the class defined by Poincare duality. It is called the Poincare dual class to S. By definition, if $\alpha \in Z^k(M)$, we have

$$\int_S \alpha = \int_M \alpha \wedge \tau.$$

One may think of τ as some kind of delta measure along S, keeping in mind however that this equation only holds true for *closed* forms α.

In fact, it is not important that S is an embedded submanifold: If S is any compact, oriented manifold of dimension $k \le m$ and $\iota \in C^\infty(S, M)$, we can define a Poincare dual class $[\tau] \in H^{m-k}_{\mathrm{comp}}(M)$ by the condition

$$\int_S \iota^* \alpha = \int_M \alpha \wedge \tau$$

for all $\alpha \in Z(M)$. Since the left side depends only on the homotopy class of ι, it follows that $[\tau]$ is invariant under homotopies of ι. Note furthermore that if $U \subset M$ is a neighborhood of the image $\iota(S)$, the Poincare dual of S in U also serves as a Poincare dual in M. In other words, one may choose the Poincare dual to be supported in arbitrary small neighborhoods of $\iota(S)$.

Example 9.5 (Poincare dual of a circle inside an annulus). The Poincare dual of $S^1 \subset \mathbb{R}^2$ inside the annulus $U = \{x \in \mathbb{R}^2 |\, 1/2 < |x| < 2\}$ is given by the 1-form τ, written in polar coordinates as

$$\tau = F'(r)\mathrm{d}r$$

where $F(r) = 0$ for $r > 1 + \epsilon$ and $F(1) = 0$ for $r < 1 - \epsilon$. To see this, we have to check the defining equation for the Poincare dual. Let $\alpha = f(r, \phi)\mathrm{d}r + g(r, \phi)\mathrm{d}\phi$ be a closed form on U. Since α is closed, $\frac{\partial f}{\partial \phi} = \frac{\partial g}{\partial r}$. We calculate, using integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}^2} \alpha \wedge \tau &= \int_{\mathbb{R}^2} g(r, \phi)F'(r)\mathrm{d}\phi \wedge \mathrm{d}r \\
&= -\int_{\mathbb{R}^2} g(r, \phi)F'(r)\mathrm{d}r \wedge \mathrm{d}\phi \\
&= \int_{\mathbb{R}^2} \frac{\partial}{\partial r}(g(r, \phi)F(r))\mathrm{d}r\mathrm{d}\phi - \int_{\mathbb{R}^2} \frac{\partial g}{\partial r}(r, \phi)F(r)\mathrm{d}r\mathrm{d}\phi \\
&= \int_{S^1} g(1, \phi)\mathrm{d}\phi - \int_{\mathbb{R}^2} \frac{\partial f}{\partial \phi}(r, \phi)F(r)\mathrm{d}\phi\mathrm{d}r \\
&= \int_{S^1} g(1 - \epsilon, \phi)\mathrm{d}\phi \\
&= \int_{r=1-\epsilon} \alpha \\
&= \int_{r=1} \alpha.
\end{aligned}$$

Note that τ is exact in $\Omega(U)$ and also in $\Omega_{\mathrm{comp}}(\mathbb{R}^2)$, since $\tau = \mathrm{d}F$.

Example 9.6 (Intersection number). Let M be a manifold, and S, S' two compact submanifolds manifolds of complementary dimension, $\dim S + \dim S' = \dim M$. Say $\dim S = p$ and $\dim S' = m - p$. Let $[\tau] \in H^{m-p}_{\mathrm{comp}}(S)$ and $[\tau'] \in H^{p}_{\mathrm{comp}}(S')$ be the two cohomology classes. The intersection number of S, S' is defined as the integral,

$$i(S, S') := \int_M \tau \wedge \tau'.$$

We will see later (?) that this is always an integer. Already at this stage, one can see that $i(S, S') = 0$ if $S \cap S' = \{0\}$. (Why?)

Example 9.7 (Linking number). Let M be any 3-manifold with $H^1(M) = 0$. (E.g., M simply connected.) Let $\gamma, \gamma' : S^1 \to M$ be two smooth loops with $\gamma(S^1) \cap \gamma'(S^1) = \emptyset$. Let U, U' be disjoint open neighborhoods of the images of γ, γ'. Let $\tau \in \Omega^2_{\mathrm{comp}}(U)$ represent the Poincare dual of γ in U and $\tau' \in \Omega^2_{\mathrm{comp}}(U')$ represent the Poincare dual of γ' in U'. Extend τ, τ' by 0 to forms on M. Since $H^2_{\mathrm{comp}}(M) = H^1(M)^* = 0$, we can write $\tau = \mathrm{d}\beta$ and $\tau' = \mathrm{d}\beta'$, where β, β' have compact (but usually not disjoint) support.

One defines the linking number of γ, γ' by the equation

$$\mathrm{lk}(\gamma, \gamma') = \int_M \beta \wedge \tau'. \tag{7}$$

Integration by parts shows that the linking number is anti-symmetric in γ, γ'. We claim that $\mathrm{lk}(\gamma, \gamma')$ is independent of all the choices involved in its definition. Suppose $\tilde{\tau}' = \tau' + \mathrm{d}\gamma$, where $\gamma \in \Omega^1_{\mathrm{comp}}(U')$. Then the right hand side of (7) does not change, since

$$\int_M \beta \wedge \mathrm{d}\gamma = -\int_M \mathrm{d}\beta \wedge \gamma = -\int_M \tau \wedge \gamma = 0$$

(since τ, γ have disjoint support.) Furthermore, the choice of U, U' does not matter, since the Poincare duals may be represented by a form supported in an arbitrarily small open neighborhood of the images of γ, γ'.

The linking number is invariant under smooth isotopies of γ, γ', provided the curves remain disjoint:

Finally, note that $\mathrm{lk}(\gamma, \gamma') = 0$ if it is possible to contract one loop to a point while keeping it disjoint from the other loop during the retraction.

Since τ, τ' have disjoint support, it follows that β is closed on the support of τ'. That is, we have

$$\mathrm{lk}(\gamma, \gamma') = \int_{S^1} (\gamma')^* \beta = -\int_{S^1} \gamma^* \beta'.$$

It turns out that the linking number is always an integer. Let us prove this for the special case $\gamma : S^1 \hookrightarrow \mathbb{R}^3$ is the unit circle in the $x - y$ plane. Let $\epsilon > 0$ be small. After applying some isotopy to γ' in $M \backslash \gamma(S^1)$, we may assume that γ' meets the region $|z| < \epsilon$ in a number of line segments, and does not meet the region $|z| < \epsilon, |r - 1| < \epsilon$. Let n_+ (resp. n_-) be the number of line segments in the region $r < 1 - \epsilon$ going from $z = -\epsilon$ to $z = \epsilon$, (resp from $z = \epsilon$ to $z = -\epsilon$). We claim that

$$\mathrm{lk}(\gamma, \gamma') = n_+ - n_-.$$

Indeed, the Poincare dual of γ may be represented by a form

$$\tau = \mathrm{d}F \wedge \mathrm{d}G = \mathrm{d}(F\mathrm{d}G)$$

where $F = F(r)$ is equal to 1 for $r < 1 - \epsilon$ and $F(r) = 0$ for $r > 1 + \epsilon$, and $G = G(z)$ is equal to 1 for $z > \epsilon$ and equal to 0 for $z < -\epsilon$. Let $\beta = F\mathrm{d}G$. Note that β is supported the region $r \le 1 + \epsilon, |z| < \epsilon$. The integral $\int_{S^1} (\gamma')^* \beta$ is a sum of integrals over the line segments described above. But $\beta = \mathrm{d}G$ on the region $U' \cap \{|r - 1| < \epsilon\}$. Thus each integral over a line segment is calculated by Stokes, and gives ± 1 depending on whether the line segment travels from $z = -\epsilon$ to $z = \epsilon$ or vice versa.

Exercise 9.8. Generalize the above concept of linking number to disjoint, immersed compact submanifolds S, S' of a manifold M, with dimensions $\dim S + \dim S' = \dim M - 1$. What are the conditions on $H(M)$ so that the definition makes sense?

10. Mapping degree

Let M, N be compact, connected, oriented manifolds of equal dimension m, and $F \in C^\infty(N, M)$ a smooth map. Let Γ_M, Γ_N be volume forms defining the orientation, normalized to have total integral equal to 1. Thus $[\Gamma_M] \in H^m(M)$ corresponds to 1 under the isomorphism $\int_M : H^m(M) \to \mathbb{R}$. The *mapping degree* of F is defined to be

$$\deg(F) = \int_N F^* \Gamma_M .$$

Basic properties of the mapping degree, which follow immediately from the definition, are: (1) $\deg(F)$ depends only on the smooth homotopy class of F. (2) or any $\omega \in \Omega^m(M)$, one has $\int_N F^*\omega = \deg(F) \int_M \omega$. (3) Under composition of maps, $\deg(F \circ G) = \deg(F)\deg(G)$. (4) If F is a diffeomorphism, then $\deg(F) = 1$ if F preserves orientation and $\deg(F) = -1$ if F reverses orientation.

Example 10.1. Let $F : S^m \to S^m$ be the diffeomorphism $x \mapsto -x$. The standard volume form Γ on S^m transforms according to $F^*\Gamma = (-1)^{m+1}\Gamma$. Thus F preserves orientation if and only if m is odd. We conclude $\deg(F) = (-1)^{m+1}$. Thus F cannot be homotopic to the identity map if m is even.

We will now show that the mapping degree is always an integer, and also give an alternative interpretation of the degree. Without proof we will use *Sard's theorem*, which implies that for every smooth map between compact manifolds, the set of regular values is open and dense. (Recall that points that are not in the image are regular values, according to our definition.) Thus let $x \in M$ be a regular value of F. Since $\dim M = \dim N$, the pre-image $F^{-1}(x)$ is zero dimensional, i.e. is a finite collection of points $y_1, \ldots, y_d$. For each $i = 1, \ldots, d$, let $\epsilon_i = \pm 1$, according to whether or not the tangent map $T_{y_i}N \to T_x M$ preserves or reverses orientation.

Theorem 10.2. *The mapping degree is given by the formula,*

$$\deg(F) = \sum_{i=1}^d \epsilon_i ,$$

for any regular value x with pre-images $y_1, \ldots, y_d$, and ϵ_i defined as above. In particular, $\deg(F)$ is an integer.

Proof. Let U be a connected open neighborhood of x, with the property that U is contained in the set of regular values of F. Let $\omega \in \Omega^m_{\mathrm{comp}}(U)$ be a form of total integral 1. Thus $\deg(F) = \int_N F^*\omega$. Choosing U sufficiently small, the pre-image $F^{-1}(U)$ is a disjoint union $\coprod_{i=1}^d V_i$ where V_i is an open neighborhood of y_i. Then F restricts to diffeomorphisms $V_i \to U$, which are orientation preserving if $\epsilon_i = 1$ and orientation reversing if $\epsilon_i = -1$. Thus

$$\deg(F) = \int_N F^*\omega = \sum_{i=1}^d \int_{V_i} (F|_{V_i})^*\omega = \sum_{i=1}^d \epsilon_i \int_U \omega = \sum_{i=1}^d \epsilon_i .$$

□

Corollary 10.3. *Suppose* $\deg(F) \neq 0$. *Then F is surjective.*

Proof. Apply the Theorem to any x in the complement of the image of F. □

Example 10.4 (Fundamental theorem of algebra). Let $p(z) = \sum_{i=0}^{d} a_i z^i$ be a polynomial of degree $d > 0$, with $a_d = 1$. The map $p : \mathbb{C} \to \mathbb{C}$ takes ∞ to ∞, so it extends to a map $F^p : S^2 \to S^2$ where we view S^2 as the one point compactification of $\mathbb{C}$. The map F^p is smooth. Replacing a_i with ta_i for $i < d$, we see that F^p is homotopic to the map defined by the polynomial $q(z) = z^d$. The equation $q(z) = 1$ has exactly d solutions, given by the dth roots of unity. Since holomorphic maps are orientation preserving at all regular points, it follows that

$$\deg(F^p) = \deg(F^q) = d.$$

Thus F^p is surjective. This shows that the equation $p(z) = a$ has at least one solution, for all $a \in \mathbb{C}$. (In fact, we see that for an open dense subset of values, it has exactly d solutions.)

Example 10.5. View S^1 as the unit circle in $\mathbb{R}^2 = \mathbb{C}$, and let the kth power map $P_k : S^1 \to S^1$ be the restriction of the map $\mathbb{C} \to \mathbb{C},\ z \mapsto z^k$. If $\Gamma = (2\pi)^{-1}\mathrm{d}\theta$ is the standard volume form on S^1, we have $P_k^*\Gamma = k\Gamma$. Thus $\deg(P_k) = k$.

Theorem 10.6. *Two maps $F_0, F_1 : S^1 \to S^1$ are smoothly homotopic if and only if they have the same mapping degree.*

Proof. Since homotopy is an equivalence relation, it suffices to show that any smooth map $F : S^1 \to S^1$ of degree k is homotopic to P_k. In fact, since $F \circ P_{-k}$ has mapping degree 0, it suffices to consider the case $k = 0$. In this case, $F^*\Gamma$ has integral 0. Locally, $F^*\Gamma = (2\pi i)^{-1}\mathrm{d}\log(F)$. But since integration is an isomorphism $H^1(S^1) \to \mathbb{R}$, we know that $F_1^*\Gamma = \mathrm{d}f$ for some globally defined function $f \in C^\infty(S^1)$. That is,

$$F(z) = e^{2\pi i f(z)}$$

for all $z \in S^1$. Let $F_t(z) := e^{2\pi i t\, f(z)}$. Then F_t defines a smooth homotopy between F_1 and $F_0 = \Pi_0$. □

11. Kuenneth formula

Suppose the manifold M is a direct product $M = B \times F$. Let π_B, π_F denote the projections from M to the two factors. The bilinear map

$$\Omega(B) \times \Omega(F) \to \Omega(M),\ (\alpha, \beta) \mapsto \pi_B^*\alpha \wedge \pi_F^*\beta$$

is a chain map, hence it induces a map in cohomology,

$$H(B) \otimes H(F) \to H(B \times F) = H(M). \tag{8}$$

Theorem 11.1 (Kuenneth theorem). *Suppose B (or F) is of finite type. Then the map* (8) *is an isomorphism. That is,*

$$H^p(B \times F) = \bigoplus_{j=0}^{p} H^j(B) \otimes H^{p-j}(F).$$

Put differently, the Kuenneth theorem says that any class in $H(B \times F)$ has a representative of the form,

$$\gamma = \sum_{i=1}^{N} \pi_B^* \alpha_i \wedge \pi_F^* \beta_i.$$

Proof. Let us temporarily introduce the notation

$$H^p(B; F) := \bigoplus_{j=0}^{p} H^j(B) \otimes H^{p-j}(F)$$

(since the following long exact sequences would get too long otherwise.) We want to show that the natural map $H^p(B; F) \to H^p(B \times F)$ is an isomorphism. The idea is once again to use induction on the number l of open sets in a good cover. Note first that the Kuenneth theorem holds for $B = \mathbb{R}^n$, since $B \times F$ retracts onto $\{0\} \times F$ in this case. Thus the Theorem is true for $l = 0$. For the induction step, we use the Mayer-Vietoris sequence. Suppose $B = U \cup V$. We have the Mayer-Vietoris sequence

$$\cdots \to H^j(U \cup V) \to H^j(U) \oplus H^j(V) \to H^j(U \cap V) \to H^{j+1}(U \cup V) \to \cdots.$$

Tensor with $H^{p-j}(F)$, and sum over j to obtain a new exact sequence,

$$\cdots \to H^p(U \cup V; F) \to H^p(U; F) \oplus H^p(V; F) \to H^p(U \cap V; F) \to H^{j+1}(U \cup V; F) \to \cdots.$$

Consider the diagram

$$\begin{array}{ccccccc}
H^p(U \cup V; F) & \longrightarrow & H^p(U; F) \oplus H^p(V; F) & \longrightarrow & H^p(U \cap V; F) & \xrightarrow[\delta]{} & H^{p+1}(U \cup V; F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^p((U \cup V) \times F) & \longrightarrow & H^p(U \times F) \oplus H^p(V \times F) & \longrightarrow & H^p((U \cap V) \times F) & \xrightarrow[\delta]{} & H^{p+1}((U \cup V) \times F)
\end{array}$$

We claim that this diagram commutes. As usual, this is fairly easy, except for the square involving the connecting homomorphism. Thus let $\alpha \in Z^j(U \cap V)$ and $\beta \in Z^{p-j}(F)$. Let χ_U, χ_V be a partition of unity for U, V, and $\pi_{U \cup V}^* \chi_U, \pi_{U \cup V}^* \chi_V$ the corresponding partition of unity for $U \times F$, $V \times F$. Then δ is realized on the level of cocycles. We have to show that the two forms

$$\delta(\pi_{U \cap V}^* \alpha \wedge \pi_F^* \beta),$$

$$\pi_{U \cup V}^* (\delta \alpha) \wedge \pi_F^* \beta$$

are cohomologous. Recall that $(\delta\alpha)|_U = \mathrm{d}(\chi_V\alpha)$, $(\delta\alpha)|_V = -\mathrm{d}(\chi_U\alpha)$ so that $\delta\alpha$ is actually supported in $U\cap V$. Similarly, $\delta(\pi^*_{U\cap V}\alpha\wedge\pi^*_F\beta)$ is supported in $(U\cap V)\times F$. We calculate, on $(U\cap V)\times F$,

$$\begin{aligned}\delta(\pi^*_{U\cap V}\alpha\wedge\pi^*_F\beta) &= \mathrm{d}(\pi^*_{U\cap V}\chi_V\alpha\wedge\pi^*_F\beta)\\ &= \pi^*_{U\cap V}\mathrm{d}(\chi_V\alpha)\wedge\pi^*_F\beta\\ &= \pi^*_{U\cup V}(\delta\alpha)\wedge\pi^*_F\beta.\end{aligned}$$

The Theorem now follows by induction on the number of elements in a good cover, together with the Five-lemma. □

As an application, we obtain the cohomology of an m-torus

$$T = \underbrace{S^1\times\cdots\times S^1}_{m\ \text{times}}.$$

As an algebra, $H(S^1)$ is an exterior algebra in one generator of degree 1. That is, $H(S^1) = \wedge\mathbb{R}$. (This just reflects the obvious fact that the volume form squares to 0.) Consequently

$$H(T) = H(S^1)\otimes\cdots\otimes H(S^1)$$

is an exterior algebra in M generators, i.e.

$$H(T) = \wedge\mathbb{R}\otimes\cdots\otimes\wedge\mathbb{R}\cong\wedge\mathbb{R}^N$$

as graded algebras.

It is convenient to introduce the following notation. For any manifold M, one calls $b_k(M) = \dim(H^k(M))$ the *kth Betti number* and one defines a polynomial in one variable t,

$$p(M)(t) = \sum_{k=0}^{\dim M} b_k(M)\,t^k,$$

called the *Poincare polynomial*. For instance, $p(\mathbb{R}^m)(t) = 1$ while $p(S^m)(t) = 1 + t^m$.

Poincare duality for a *compact* connected oriented m-dimensional manifold M says that

$$t^m p(M)(t^{-1}) = p(M)(t),$$

and the Kuenneth formula implies that

$$p(M_1\times M_2) = p(M_1)p(M_2).$$

For instance, the Poincare polynomial for an m-torus $T = S^1\times\cdots\times S^1$ is $p(T)(t) = (1+t)^m$.

There is also a Kuenneth theorem for compactly supported cohomology. It can be derived using the Mayer-Vietoris sequence for compactly supported cohomology, or simply by Poincare duality. One introduce Betti numbers $b_k(M)_{\mathrm{comp}}$ and a Poincare polynomial $p(M)_{\mathrm{comp}}$ as before.

12. De Rham theorem

12.1. Cech cohomology. Let A be an abelian group, written additively. We are mostly interested in the cases $A = \mathbb{R}, \mathbb{Z}, \mathbb{Z}_p$. Let M be a manifold, and $\mathcal{U} = \{U_\alpha\}$ an open cover of M. For any collection of indices $\alpha_0, \ldots, \alpha_p$ such that $U_{\alpha_0} \cap \ldots \cap U_{\alpha_p} \neq \emptyset$, let

$$U_{\alpha_0 \ldots \alpha_p} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$$

One defines the *Cech complex*

$$\check{C}^0(\mathcal{U}, A) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, A) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, A) \xrightarrow{\delta} \cdots$$

as follows. A Cech-p-cochain $f \in \check{C}^p(\mathcal{U}, A)$ is a function

$$f = \coprod f_{\alpha_0 \ldots \alpha_p} : \coprod_{\alpha_0 \ldots \alpha_p} U_{\alpha_0 \ldots \alpha_p} \to A,$$

where each $f_{\alpha_0 \ldots \alpha_p} : U_{\alpha_0 \ldots \alpha_p} \to A$. is locally constant, and anti-symmetric in its indices. The differential is defined by the formula,

$$(\delta f)_{\alpha_0, \ldots, \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{p+1}}$$

where the hat means that the entry is to be omitted. For example, if $f \in \check{C}^0(\mathcal{U}, A)$,

$$(\delta f)_{\alpha_0 \alpha_1} = f_{\alpha_1} - f_{\alpha_0}$$

if $g \in \check{C}^1(\mathcal{U}, A)$,

$$(\delta g)_{\alpha_0 \alpha_1 \alpha_2} = g_{\alpha_1 \alpha_2} - g_{\alpha_0 \alpha_2} + g_{\alpha_0 \alpha_1}.$$

We thus see that

$$\delta(\delta f) = (f_{\alpha_2} - f_{\alpha_1}) - (f_{\alpha_2} - f_{\alpha_0}) + (f_{\alpha_1} - f_{\alpha_0}) = 0.$$

Exercise 12.1. Verify that $\delta \circ \delta = 0$ in general.

The cohomology groups $\check{H}^p(\mathcal{U}, A) := H^p(\check{C}(\mathcal{U}, A))$ are called *Cech cohomology groups* with coefficients in A.

A nice feature of the Cech cohomology groups is that they are purely combinatorial, reflecting the intersections of elements in our open cover. On the other hand, if we want to use them to define invariants of a manifold, there is a problem that they do depend on the cover.

If $A \to A'$ is a homomorphism of abelian groups, one has an induced homomorphism of Cech cochain, and hence a homomorphism of cohomology groups, $\check{H}^p(\mathcal{U}, A) \to \check{H}^p(\mathcal{U}, A')$.

Example 12.2. Suppose each U_α is connected. A Cech 0-cocycle is just a collection of elements $f_\alpha \in A$ that agree on overlaps. That is, if M is connected, $\check{H}^0(\mathcal{U}, A) = A$.

Example 12.3. Suppose M is connected and all U_α are diffeomorphic to $\mathbb{R}^m$. If $\omega \in Z^1(M)$, we can choose $\nu_\alpha \in \Omega^0(U_\alpha)$ such that $\mathrm{d}\nu_\alpha = \omega|_{U_\alpha}$. Then $f_{\alpha\beta} := \nu_\beta - \nu_\alpha$ is a constant $f_{\alpha\beta} \in \mathbb{R}$, and clearly defines a 1-cocycle in $\check{C}^0(\mathcal{U}, \mathbb{R})$. A different choice ν'_α differs from ν_α by a constant c_α, thus $f_{\alpha\beta} = c_\beta - c_\alpha = (\delta c)_{\alpha\beta}$. Hence, the Cech cohomology class of $f_{\alpha\beta}$ does not change. Similarly, if ω is cohomologous to ω', the difference is the differential $\mathrm{d}g$ of a global function $g \in \Omega^0(M)$. But replacing ν_α by $\nu'_\alpha = \nu_\alpha + g|_{U_\alpha}$, does not change $f_{\alpha\beta}$. The upshot is that there is a well-defined map

$$H^1(M) \to \check{H}^1(\mathcal{U}, \mathbb{R}).$$

Suppose $[\omega]$ is in the kernel of this map. This means that the ν_α can be chosen such that $f_{\alpha\beta}$ becomes a coboundary, i.e. $f_{\alpha\beta} = c_\beta - c_\alpha$ for some constants c_α. Replacing ν_α by $\nu_\alpha - c_\alpha$ takes $f_{\alpha\beta}$ to 0. That is the new functions ν_α agree on overlaps, and define a function $\nu \in \Omega^0(M)$ with $\mathrm{d}\nu = \omega$. Thus $[\omega] = 0$. This shows that the map $H^1(M) \to \check{H}^1(\mathcal{U}, \mathbb{R})$ is injective. Conversely, suppose $f_{\alpha\beta}$ is a given Cech 1-cocycle. Choose a partition of unity χ_α, and define $\nu_\alpha = \sum_\gamma \chi_\gamma f_{\alpha\gamma}$. Then

$$\nu_\beta - \nu_\alpha = \sum_\gamma \chi_\gamma (f_{\beta\gamma} - f_{\alpha\gamma}) = \sum_\gamma \chi_\gamma f_{\beta\alpha} = f_{\beta\alpha}$$

and therefore

$$\mathrm{d}\nu_\beta - \mathrm{d}\nu_\alpha = 0$$

Hence, there is a unique form $\omega \in Z^1(M)$ with $\omega|_{U_\alpha} = \mathrm{d}\nu_\alpha$. This shows that the map $H^1(M) \to \check{H}^1(\mathcal{U}, \mathbb{R})$ is an isomorphism.

12.2. De Rham's theorem.

Theorem 12.4 (De Rham). *If $\mathcal{U}$ is a good cover, the Cech cohomology groups $\check{H}^p(\mathcal{U}, \mathbb{R})$ are canonically isomorphic to the de Rham cohomology groups $H^p(M)$. In particular, they are independent of the choice of good cover.*

Note that in the case of a good cover, the open sets are in particular connected. Thus the locally constant functions $f_{\alpha_0 \ldots \alpha_p}$ are really just elements of the group A.

Below we will present A. Weil's proof of de Rham's theorem. A first step is to introduce, for each q, another type of Cech complex

$$\check{C}^0(\mathcal{U}, \Omega^q) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \Omega^q) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \Omega^q) \xrightarrow{\delta} \cdots$$

as follows. A Cech-p-cochain $\omega \in \check{C}^p(\mathcal{U}, \Omega^q)$ is a collection of q-forms $\omega_{\alpha_0 \ldots \alpha_p} \in \Omega^q(U_{\alpha_0 \ldots \alpha_p})$, anti-symmetric in its indices. The differential is defined as before. We define $\check{H}^p(\mathcal{U}, \Omega^q) := H^p(\check{C}(\mathcal{U}, \Omega^q))$.

Theorem 12.5. *Suppose $\mathcal{U}$ is a good cover. Then $\check{H}^p(\mathcal{U}, \Omega^q) = 0$ for all $p > 0$.*

Proof. Let χ_α be a partition of unity subordinate to the given cover. Define an operator

$$h : \check{C}^p(\mathcal{U}, \Omega^q) \to \check{C}^{p-1}(\mathcal{U}, \Omega^q)$$

by

$$(h\omega)_{\alpha_0\ldots\alpha_{p-1}} = \sum_\alpha \chi_\alpha \omega_{\alpha\alpha_0\ldots\alpha_{p-1}}$$

We verify that h is a homotopy operator: Indeed

$$\delta(h\omega)_{\alpha_0\ldots\alpha_p} = \sum_{i,\alpha}(-1)^i \chi_\alpha \omega_{\alpha\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p-1}}$$

and

$$\begin{aligned} h(\delta\omega)_{\alpha_0\ldots\alpha_p} &= \sum_\alpha \chi_\alpha(\delta\omega)_{\alpha\alpha_0\ldots\alpha_p} \\ &= \sum_\alpha \chi_\alpha \omega_{\alpha_0\ldots\alpha_p} - \sum_i (-1)^i \chi_\alpha \omega_{\alpha\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_p}, \end{aligned}$$

thus $h\delta + \delta h = \mathrm{Id}$ on forms of degree $p > 0$. □

We now arrange $C^{p,q} := \check{C}^p(\mathcal{U}, \Omega^q)$ as a *double complex*, as follows:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
\check{C}^0(\mathcal{U},\Omega^2) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^2) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^2) & \longrightarrow & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\check{C}^0(\mathcal{U},\Omega^1) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^1) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^1) & \longrightarrow & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\check{C}^0(\mathcal{U},\Omega^0) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^0) & \xrightarrow[\delta]{} & \check{C}^1(\mathcal{U},\Omega^0) & \longrightarrow &
\end{array}$$

Notice that the cohomology groups are trivial in both horizontal and vertical directions, except in degree 0. One can make the double complex into a single complex, by letting

$$K^p := \bigoplus_{j=0}^p \check{C}^j(\mathcal{U}, \Omega^{p-j}),$$

with differential, $D = \delta + (-1)^p \mathrm{d}$. The factor $(-1)^p$ is necessary so that $D^2 = 0$. One calls $H(K, D) = H(\check{C}(\mathcal{U}, \Omega), D)$ the *total cohomology* of the double complex. A D-cocycle is a collection $(\nu^{(0)}, \ldots, \nu^{(p)})$ of cochains $\nu^{(j)} \in \check{C}^j(\mathcal{U}, \Omega^{p-j})$ with $D(\nu^{(0)} + \ldots + \nu^{(p)}) = 0$.

This gives a set of equations,

$$\begin{aligned}
\mathrm{d}\nu^{(0)} &= 0 \\
\mathrm{d}\nu^{(1)} &= (-1)^p \delta\nu^{(0)} \\
\mathrm{d}\nu^{(2)} &= (-1)^{p-1} \delta\nu^{(1)} \\
\cdots & \quad \cdots \\
\mathrm{d}\nu^{(p)} &= -\delta\nu^{(p-1)}, \\
0 &= \delta\nu^{(p)}
\end{aligned}$$

LEMMA 12.6. *The restriction map $\Omega^p(M) \to \check{C}^0(\mathcal{U}, \Omega^p) \subset K^p$ induces an isomorphism in cohomology,*

$$H^p(M) = H^p(\check{C}(\mathcal{U}, \Omega), D).$$

PROOF. Surjectivity: Let $(\nu^{(0)}, \ldots, \nu^{(p)})$ be a D-cocycle. We have to show that it is cohomologous to a cocycle in the image of the restriction map $\Omega^p(M) \to \check{C}^0(\mathcal{U}, \Omega^p)$. Since $\delta\nu^{(p)} = 0$ and the δ-cohomology is trivial, we can write $\nu^{(p)} = \delta\beta^{(p-1)}$, where $\beta^{(p-1)} \in \check{C}^{p-1}(\mathcal{U}, \Omega^0)$. Subtracting from $(\nu^{(0)}, \ldots, \nu^{(p)})$ the coboundary $D\beta^{(p-1)}$, we thus achieve $\nu^{(p)} = 0$. Then $\delta\nu^{(p-1)} = \pm\mathrm{d}\nu^{(p)} = 0$. Thus we can write $\nu^{(p-1)} = \delta\beta^{(p-2)}$, and subtracting $D\beta^{(p-2)}$ we achieve $\nu^{(p-1)} = 0$. Proceeding in this manner, we successively subtract D-coboundaries and arrange that $\nu^{(j)} = 0$ for all $j > 0$. The remaining $\nu^{(0)} \in \check{C}^0(\mathcal{U}, \Omega^0)$ satisfies $\mathrm{d}\nu^{(0)} = 0$ and $\delta\nu^{(0)} = 0$. Thus it is the restriction of a closed form $\omega \in \Omega^p(M)$.

Injectivity: Let $\omega \in Z^p(M)$ be a cocycle, and $\nu^{(0)} \in \check{C}^0(\mathcal{U}, \Omega^p)$ be defined by restriction. Suppose $\nu^{(0)}$ is the D-coboundary of some $(\beta^{(0)}, \ldots, \beta^{(p-1)}) \in K^{p-1}$. Then $\delta\beta^{(p-1)} = 0$, so by adding a D-coboundary we can arrange $\beta^{(p-1)} = 0$. Proceeding in this manner, we can arrange that all $\beta^{(j)} = 0$, except maybe $\beta^{(0)}$. The remaining form $\beta^{(0)}$ satisfies $\delta\beta^{(0)} = 0$, which means that it is the restriction of a global form γ, and $(-1)^p \mathrm{d}\beta^{(0)} = \nu^{(0)}$, meaning that $(-1)^p \mathrm{d}\gamma = \omega$. Thus $[\omega] = 0$ and we are done. □

Notice that the proof was purely algebraic. It involved that the δ-cohomology is trivial in positive degree, and that the kernel of the map $\delta : \check{C}^0(\mathcal{U}, \Omega^q) \to \check{C}^1(\mathcal{U}, \Omega^q)$ is exactly the image of the map $\Omega^q(M) \to \check{C}^0(\mathcal{U}, \Omega^q)$.

Hence, the proof works equally well with the roles of δ, d reversed. That is, we also have

LEMMA 12.7. *The restriction map $\check{C}^p(\mathcal{U}, \mathbb{R}) \to \check{C}^p(\mathcal{U}, \Omega^0) \subset K^p$ induces an isomorphism in cohomology,*

$$\check{H}^p(\mathcal{U}, \mathbb{R}) = H^p(\check{C}(\mathcal{U}, \Omega), D).$$

Putting the two Lemmas together, we have proved de Rham's theorem:

$$\check{H}^p(\mathcal{U}, \mathbb{R}) = H^p(\check{C}(\mathcal{U}, \Omega), D) = H^p(M).$$

As one application, one can introduce the notion of integral de Rham classes: Indeed, the inclusion $\mathbb{Z} \to \mathbb{R}$ induces a homomorphism of Cech cochains, hence a map in cohomology $\check{H}^p(\mathcal{U}, \mathbb{Z}) \to \check{H}^p(\mathcal{U}, \mathbb{R}) = H^p(M)$. Classes in the image of this map are called integral.

One important consequence of de Rham's theorem is that they are *topological invariants* (since the Cech cohomology groups are). Hence they cannot be used to distinguish different manifold structures on a given topological space.

12.3. Relative cohomology. Let $\Phi \in C^\infty(M, N)$ be a smooth map. Define a complex

$$\Omega^p(\Phi) := \Omega^{p-1}(M) \oplus \Omega^p(N)$$

with differential, $\mathrm{d}(\alpha, \beta) = (\Phi^*\beta - \mathrm{d}\alpha, \mathrm{d}\beta)$. It is straightforward to check that d squares to 0. A class in $H^p(\Phi)$ is called a relative de Rham cohomology class for the map Φ. Thus classes in $H^p(M)$ are represented by closed p-forms β on N together with a primitive $\alpha \in \Omega^{p-1}(M)$ for the pull-back $\Phi^*\beta$. The sequence

$$0 \longrightarrow \Omega^{p-1}(M) \longrightarrow \Omega^p(\Phi) \longrightarrow \Omega^p(N) \longrightarrow 0,$$

where the first map takes α to $(-1)^p(\alpha, 0)$ and the second map takes $(0, \beta)$ to β, is an exact sequence of differential complexes. Hence it induces a long exact sequence in cohomology:

$$\longrightarrow H^{p-1}(M) \longrightarrow H^p(\Phi) \longrightarrow H^p(N) \longrightarrow H^p(M) \longrightarrow \cdots$$

An immediate consequence is that if M is contractible, then $H^p(\Phi) = H^p(N)$ for $p > 0$, while if N is contractible, then $H^p(\Phi) = H^{p-1}(M)$ for $p > 0$.

Suppose A is an abelian group, and U_i, V_i are good covers of M, N such that $\Phi(U_i) \subset V_i$. Such covers exist: Start with a good cover V_i of N, and replace the cover $\Phi^{-1}(V_i)$ by a good refinement U_i. (We may use the same index set, if we allow the same V_i to appear several times.) Let $\check{C}^\bullet(M, A), \check{C}^\bullet(N, A)$ be the Cech complexes for the given covers. We then have a pull-back map $\Phi^* : \check{C}^\bullet(N, A) \to \check{C}^\bullet(M, A)$ which one can use to define

$$\check{C}^p(\Phi, A) = \check{C}^{p-1}(M, A) \oplus \check{C}^p(N, A)$$

with differential

$$\delta(\mu, \nu) = (\Phi^*\nu - \delta(\mu), \delta(\nu)).$$

Again, we have a long exact sequence

$$\longrightarrow \check{H}^{p-1}(M, A) \longrightarrow \check{H}^p(\Phi, A) \longrightarrow \check{H}^p(N, A) \longrightarrow \check{H}^p(M, A) \longrightarrow \cdots$$

De Rham's theorem extends to relative cohomology:

THEOREM 12.8. *There is a canonical isomorphism $H^p(\Phi) \cong \check{H}^p(\Phi, \mathbb{R})$.*

PROOF. As in the proof of de Rham's theorem, it is useful to introduce auxiliary Cech cohomology groups. Consider the double complex

$$C^{p,q} := \check{C}^p(\Phi, \Omega^q) = \check{C}^{p-1}(M, \Omega^q) \oplus \check{C}^p(N, \Omega^q)$$

with the obvious differentials d, δ. Since we are working with good covers, and since the complex $\Omega^p(\Phi)$ is acyclic if M, N are contractible, it follows that the columns of the double complex are acyclic. We claim that the rows are acyclic as well. To this end, we need to generalize Theorem 12.5 to our setting. Indeed, let h_N, h_M denote the homotopy operators for the Cech complexes of M, N, cf. Theorem 12.5. Then

$$h:\, \check{C}^p(\Phi, \Omega^q) \to \check{C}^{p-1}(\Phi, \Omega^q), \quad h(\alpha, \beta) = (h_M(\Phi^*(h_N\beta) - \alpha), h_N\beta)$$

is a homotopy operator for the complex $\check{C}^\bullet(\Phi, \Omega^q)$. We verify:

$$\begin{aligned} h\delta(\alpha,\beta) &= h(\Phi^*\beta - \delta\alpha, \delta\beta) \\ &= (h_M(\Phi^*(h_N\delta\beta) - \Phi^*\beta + \delta\alpha), h_N\delta\beta) \\ &= (h_M(-\Phi^*\delta(h_N\beta) + \delta\alpha), h_N\delta\beta) \\ &= (h_M\delta(\alpha - \Phi^*(h_N\beta)), h_N\delta\beta) \end{aligned}$$

while

$$\begin{aligned} \delta h(\alpha,\beta) &= \delta(h_M(\Phi^*(h_N\beta) - \alpha), h_N\beta) \\ &= (\Phi^*(h_N\beta) + \delta h_M(\alpha - \Phi^*(h_N\beta)), \delta h_N\beta). \end{aligned}$$

Adding the two expressions, we obtain

$$h\delta(\alpha,\beta) + \delta h(\alpha,\beta) = (\alpha,\beta)$$

as desired. Hence the proof of de Rham's theorem goes through as before. □

13. Fiber bundles

13.1. Fiber bundles and vector bundles. In first approximation, a fiber bundle with fiber F is a smooth map $\pi:\, E \to B$ with fibers diffeomorphic to a given fiber F, in such a way that E is locally the product of the base and the fiber:

DEFINITION 13.1. A *fiber bundle over with standard fiber* F is a manifold E (called the *total space*) together with a map $\pi:\, E \to B$ to another manifold B (called the *base*), with the following property: There exists an open covering U_α of B, and diffeomorphisms

$$\Psi_\alpha:\, \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

such that

$$\mathrm{pr}_{U_\alpha} \circ \Psi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}.$$

A smooth map $\sigma:\, B \to E$ is called a *section* of E if $\pi \circ \sigma = \mathrm{Id}_B$. The space of sections is denoted $\Gamma^\infty(B, E)$.

Note that it follows from the definition that π is smooth and that it is a surjective submersion. The maps Ψ_α are called *local trivializations* of the fiber bundle.

For every subset $S \subset B$, we denote $E_S = \pi^{-1}(S)$. In particular, taking S to be a point $b \in B$, E_b is the fiber $\pi^{-1}(b)$. If $U \subset B$ is an open neighborhood, E_U is again a fiber bundle.

Definition 13.2. Two fiber bundles $\pi : E \to B$ and $\pi' : E' \to B$ with standard fiber F are called *isomorphic* if there exists a diffeomorphism $\Psi : E \to E'$ such that $\pi' \circ \Psi = \pi$.

The direct product $E = B \times F$ with $\pi : E \to B$ the projection to the first factor, is a fiber bundle called the *trivial bundle*. A bundle is called trivializable if it isomorphic to the trivial bundle. Clearly, existence of sections is a necessary (but not sufficient) condition for a fiber bundle to be isomorphic to the trivial bundle.

Exercise 13.3. a) View $S^1 \subset \mathbb{R}^2 = \mathbb{C}$ as the unit circle in the complex plane, with coordinates $z \in \mathbb{C}$. Let k be a non-zero integer. The map $\mathbb{C} \to \mathbb{C},\ z \mapsto z^k$ restricts to a map $\pi : S^1 \to S^1$. Construct local trivializations to show that π is non-trivial fiber bundle with fiber $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

b) Classify the fiber bundles $\pi : E \to S^1$ with fiber $\mathbb{Z}_2$, up to isomorphism. How about fiber $\mathbb{Z}_k$ for arbitrary integers $k \geq 2$?

Exercise 13.4. Show that the map $\pi : S^m \to \mathbb{R}P(m)$ makes S^m into a fiber bundle with fiber $\mathbb{Z}_2$.

Exercise 13.5 (Hopf fibration). Complex projective space may e defined as a quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the relation, $z' \sim z$ if and only if $z' = cz$, where $c \in S^1 \subset \mathbb{C}$ is a complex number of absolute value 1. Let $\pi : S^{2n+1} \to \mathbb{C}P(n)$ be the quotient map. Show that that π is a fiber bundle with fiber $F = S^1$. For $n = 1$, this becomes a fiber bundle $\pi : S^3 \to S^2 = \mathbb{C}P(1)$ called the *Hopf fibration*.

Lemma 13.6. *If B is smoothly contractible, then any fiber bundle E over B is isomorphic to the trivial bundle.*

Sketch of proof. Let $S : I \times B \to B$ be a smooth retraction onto $b \in B$. Let $F = E_b$ be the fiber over b. Given a Riemannian metric on E, there exists a unique smooth lift $\hat{S} : I \times E \to E$ such that for all $t \in I$,

$$\pi(\hat{S}(t, x)) = S(t, \pi(x))$$

and such that

$$\frac{\partial \hat{S}}{\partial t}(t, x) \in (T_x F_{\pi(x)})^{\perp}.$$

For each $y \in B$, the restriction of $\hat{S}$ to $\pi^{-1}(y) \times \{1\}$ is a diffeomorphism $E_y \to E_b = F$. The map

$$\Psi : E \to B \times F, \quad x \mapsto (\pi(x), \hat{S}(x, 1))$$

is the desired isomorphism with the trivial bundle. □

Definition 13.7. Let $\pi : E \to B$ be a fiber bundle. A smooth map $\sigma : B \to E$ is called a *section* of π if $\pi \circ \sigma = \mathrm{id}_B$. The set of sections is denoted $\Gamma^\infty(B, E)$.

13.2. Vector bundles. A vector bundle is a fiber bundle whose fibers have the structure of vector spaces. More precisely:

DEFINITION 13.8. Let M be a manifold. A *real (resp. complex) vector bundle of rank* k *over* M is a fiber bundle $\pi : E \to M$ with standard fiber $F = \mathbb{R}^k$ (resp. $\mathbb{C}^k$), such that all fibers E_x ($x \in M$) carry the structure of real (resp. complex) vector spaces, and such that the local trivializations $E_{U_\alpha} \cong U_\alpha \times F$ can be chosen fiberwise $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear). An *isomorphism of vector bundles* $\pi : E \to M$ and $\pi' : E' \to M$ is a fiber bundle isomorphism that is fiberwise linear.

One can think of a vector bundle as a family of vector spaces smoothly parametrized by the base. Suppose E is a real vector bundle with local trivializations $\Psi_\alpha : E_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$. On $U_{\alpha\beta}$ the map

$$\Psi_\alpha \circ \Psi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$$

has the form

$$\Psi_\alpha \circ \Psi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x).v)$$

where $g_{\alpha\beta}(x) \in \mathrm{GL}(k, \mathbb{R})$, the group of invertible $k \times k$-matrices. The *transition functions* $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})$ have the cocycle property,

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_{\alpha\beta\gamma}$. Conversely, given any cover U_α of a manifold M, and any collection of functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})$ with the cocycle property, there is a unique vector bundle having the $g_{\alpha\beta}$'s as transition functions. Indeed, E may be defined as a quotient

$$E = \coprod_\alpha (U_\alpha \times \mathbb{R}^k)/\sim$$

where for any $x \in U_{\alpha\beta}$ and $v \in \mathbb{R}^k$, $(x, v) \in U_\beta \times \mathbb{R}^k$ is declared equivalent to $(x, g_{\alpha\beta}(v)) \in U_\alpha \times \mathbb{R}^k$.

13.3. Examples of vector bundles.

EXAMPLES 13.9.

(a) Let $M = S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Define $E \subset S^n \times \mathbb{R}^{n+1}$ to be the set of pairs (x, v) with $x \cdot v = 0$. Then E is a vector bundle of rank n. The map $(x, v) \mapsto (-x, -v)$ is a vector bundle homomorphism.

(b) Let $M = \mathbb{R}P(n)$. Each point $x \in \mathbb{R}P(n)$ represents a 1-dimensional subspace of $E_x \subset \mathbb{R}^n$. Let

$$E \subset \mathbb{R}P(n) \times \mathbb{R}^{n+1}$$

be the set of all (x, v) such that v is in the 1-dimensional subspace determined by x. Then E is a vector bundle of rank 1 (also called a (real) line bundle). It is called the *tautological bundle* over projective space.

(c) This example generalizes: Let $M = \mathrm{Gr}_{\mathbb{R}}(k,n)$ be the Grassmannian of k-planes in $\mathbb{R}^n$. By definition, each $x \in M$ represents a k-dimensional subspace of $E_x \subset \mathbb{R}^n$. Let

$$E \subset \mathrm{Gr}_{\mathbb{R}}(k,n) \times \mathbb{R}^n$$

be the set of al (x,v) such that v is contained in the k-dimensional subspace parametrized by x. The total space of this vector bundle is called the *Stiefel manifold*.

(d) Let M be a manifold with atlas $\{(U_\alpha, \phi_\alpha)\}$. For each $U_{\alpha\beta}$, the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha\beta}) \to \phi_\alpha(U_{\alpha\beta})$ is a diffeomorphism. For $x \in U_{\alpha\beta}$ let $g_{\alpha\beta}(x) \in \mathrm{GL}(k,\mathbb{R})$ be the Jacobian of the map $\phi_\alpha \circ \phi_\beta^{-1}$ at $\phi_\beta(x)$. Clearly, the $g_{\alpha\beta}$ satisfy the cocycle condition. The resulting vector bundle is the *tangent bundle* TM of M. Its fibers $(TM)_x$ are canonically identified with the tangent spaces T_xM. (Recall that any choice of chart identifies $T_xM \cong \mathbb{R}^k$, and a change of chart changes the identification by the Jacobian of the change of coordinates.) The space of sections is just the space of vector fields:

$$\mathfrak{X}(M) = \Gamma^\infty(M, TM).$$

Note that it seems somewhat miraculous from this perspective that $\mathfrak{X}(M)$ carries a natural Lie bracket.

(e) Let S be an embedded submanifold of a manifold M. Then the restriction $TM|_S$ is a vector bundle over S, containing TS. The quotient bundle $N_S := TM|_S/TS$ with fibers $(N_S)_x := T_xM/T_xS$ is again a vector bundle called the normal bundle of S in M.

All standard constructions for vector spaces carry over to vector bundles. Thus, if $E \to M$ is a vector bundle, one can form the *dual bundle* $E^* \to M$ with fibers $(E^*)_x = E_x^*$, and the exterior powers $\wedge^p E \to M$ with fibers $(\wedge^p E)_x = \wedge^p E_x$. If $E' \subset E$ is a vector subbundle, one can form the *quotient bundle* E/E' with fibers $(E/E')_x = E_x/E'_x$. Similarly, one defines *tensor products*, *direct sums*, ...

Exercise 13.10. Work out the details of all these claims. E.g., show that the disjoint union $E^* := \coprod_{x\in M} E_x^*$ carries a natural structure of a vector bundle. What are the transition functions?

Starting from the tangent bundle TM one can form the dual bundle $T^*M = (TM)^*$, with sections $\Gamma^\infty(M, T^*M) = \Omega^1(M)$. One can also form $\wedge^k T * M$, with sections $\Gamma^\infty(M, \wedge^k T^*M) = \Omega^k(M)$. If $S \subset M$ is any submanifold, one can consider the restriction $TM|_S \to S$; sections of $TM|_S$ are called *vector fields along* S. The quotient bundle $N_S := TM|_S/TS$ is called the *normal bundle to S in M*. Given a Riemannian metric on M, one identifies N_S with the orthogonal complement of TS in $TM|_S$, that is one has a splitting $TM|_S = TS \oplus N_S$.

We will need the following fact about the normal bundle.

THEOREM 13.11. *Suppose M is a manifold, S a compact embedded submanifold, and N_S the normal bundle of S inside M. Then there exist open neighborhoods V of S in M and U of S in N_S, and a diffeomorphism $\Psi : U \to V$ such that Ψ restricts to the identity map on S.*

That is, the normal bundle N_S serves as a "local model" for a neighborhood of S in M.

SKETCH OF PROOF. Put a Riemannian metric on M. Suppose for simplicity that M is geodesically complete. (That is, all geodesics exist for all time.) Then N_S gets identified with the set of all vectors in TM_S that are orthogonal to TS. Geodesic flow defines a map $N_S \times \mathbb{R} \to M$. Let Ψ be the restriction of this map to $NS \times \{1\}$. One verifies that the tangent map along $S \subset N_S$ is invertible , so by the implicit function theorem Ψ is a diffeomorphism on a neighborhood U of S. □

14. The Thom class

14.1. Thom isomorphism. Let $\pi : E \to M$ be a rank k vector bundle over a compact manifold M of dimension m. Since E retracts onto M, the pull-back map induces an isomorphism

$$\pi^* : H^p(M) \cong H^p(E).$$

Let us suppose that both M and E carry orientations. Then Poincare duality tells us that we have a dual isomorphism,

$$H^q_{\mathrm{comp}}(E) \to H^{q-k}(M).$$

where $q = m + k - p$. This isomorphism has a simple meaning: Suppose $\beta \in Z(M)$ and $\alpha \in Z_{\mathrm{comp}}(E)$. Then

$$\int_E \pi^*\beta \wedge \alpha = \int_M \pi_*(\pi^*\beta \wedge \alpha) = \int_M \beta \wedge \pi_*\alpha,$$

where π_* is integration over the fibers.

THEOREM 14.1 (Thom isomorphism). *Let $\pi : E \to M$ be a rank k vector bundle over a compact manifold M, with a fiberwise orientation. Then integration over the fibers defines an isomorphism,*

$$\pi_* : H^p_{\mathrm{comp}}(E) \to H^{p-k}(M).$$

REMARK 14.2. The argument given above only applies to the case where M is also oriented (and E carries the induced orientation.) However, one can show that this assumption is not necessary. One can also drop the assumption that M be compact, if one replaces $H^p_{\mathrm{comp}}(E)$ with the cohomology of the complex of forms with compact support *in fiber direction.*

In particular, there is a unique class $[\mathrm{Th}(E)] \in H^k_{\mathrm{comp}}(E)$, called the *Thom class*, with property

$$\pi_* \mathrm{Th}(E) = 1.$$

A representative of the Thom class is called a Thom form. Now let $\iota : M \to E$ be the inclusion of the zero section.

LEMMA 14.3. *The Thom class has the property*

$$[\pi_*(\mathrm{Th}(E) \wedge \alpha)] = [\iota^* \alpha]$$

for all $[\alpha] \in H(M)$.

PROOF. Let $\beta = \iota^*\alpha$. The inclusion ι is a homotopy inverse to the projection π, since $\pi \circ \iota = \mathrm{Id}_M$ and since $\iota \circ \pi$ is homotopic to Id_E. Therefore, the pull-back maps ι^*, π^* are inverse to each other in cohomology, and $[\alpha] = [\pi^*\beta]$. We have

$$\pi_*(\mathrm{Th}(E) \wedge \pi^*\beta) = \pi^* \mathrm{Th}(E) \wedge \beta = \beta,$$

by the properties of the fiber integration map. □

REMARK 14.4. The definition of the Thom for extends to non-compact manifolds M that need not be orientable. All that is required is a fiberwise orientation of the vector bundle $\pi : E \to M$. A differential form $\omega \in \Omega(E)$ is said to have fiberwise compact support if for all compact subsets $K \subset M$, the intersection $\pi{-}1(K) \cap \mathrm{supp}(\omega)$ is compact. The space $\Omega_{f.c.}(E)$ of differential forms with fiberwise compact support is a differential complex, and one defines the corresponding cohomology with fiberwise compact support $H_{f.c.}(E)$. One can prove that the fiber integration map defines an isomorphism $\pi_* : H_{f.c.}(E) \to H(M)$, and one defines the Thom class to be the inverse image of $1 \in H^0(M)$ under this isomorphism. The space $\Omega_{f.c.}(E)$ is a module for $\Omega(E)$ under wedge product, and in cohomology one once again has Lemma 14.3.

DEFINITION 14.5. The class $\mathrm{Eul}(E) := \iota^* \mathrm{Th}(E) \in H^k(M)$ is called the Euler class of the oriented rank k vector bundle E. If M be a compact oriented manifold, one calls $\mathrm{Eul}(TM)$ the Euler class of the manifold M.

Note that $e(M) = 0$ if the tangent bundle can be trivialized.

DEFINITION 14.6. Suppose $K = \bigoplus_{p \geq 0} K^p$ is a differential complex with finite dimensional cohomology $H(K) = \bigoplus_{p \in \mathbb{Z}} H^p(K)$. The polynomial $p(K)(t) = \sum_p t^p \dim H^p(K)$ is called the *Poincare polynomial* of (K, d), and $p(K)(-1) \in \mathbb{Z}$ is the *Euler characteristic*. In particular, if $K = \Omega(M)$, one calls $p(M) := p(\Omega(M))$ the Poincare polynomial of M and $p(M)(-1)$ the Euler characteristic of M.

THEOREM 14.7 (Gauss-Bonnet-Chern). *Let M be a compact oriented manifold. Then the Euler characteristic of M equals the integral of its Euler class,*

$$\int_M \mathrm{Eul}(TM) = \sum_i (-1)^i \dim H^i(M).$$

In particular, this integral is an integer.

Theorem 14.7 is quite remarkable, in that it expresses a topological quantity (the alternating sum of de Rham cohomology groups) in local terms, as the integral of a differential form over the manifold.

REMARKS 14.8. a) The full Gauss-Bonnet-Chern theorem gives, in addition, a concrete prescription how to represent $\mathrm{Eul}(TM)$ in terms of "curvature invariants". (The classical Gauss-Bonnet theorem says that for any compact connected oriented 2-manifold Σ, the integral of the so-called Gauss curvature equals the Euler characteristic $2-2g$, where g is the genus = number of handles.

b) Of course, $\mathrm{Eul}(TM) = 0$ whenever the tangent bundle of M is trivial. For instance, it can be shown (see e.g. the book on characteristic classes by Milnor-Stasheff) that the tangent bundle of any compact oriented 3-manifold is trivial.

c) Note that $p_M(-1) = 0$ if $\dim M$ is odd. This follows by putting $t = -1$ in the equation for Poincare duality, $p_M(t^{-1}) = t^{-m} p_M(t)$.

The proof of Theorem 14.7 requires some preparation. Let $\iota : \Delta \subset M \times M$ be the diagonal, and $[\tau_\Delta]$ its Poincare dual. Now let $N_\Delta \to \Delta$ denote the normal bundle of Δ in $M \times M$.

LEMMA 14.9. *There is an isomorphism* $\iota^* N_\Delta = T\Delta$.

PROOF. Choose a Riemannian metric on M, and let $M \times M$ be equipped with the product metric. For any $x \in M$, the tangent space to Δ at $\iota(x)$ consists of vectors (v, v), and its normal space of vectors $(v, -v)$ where $v \in T_x M$. □

By the tubular neighborhood theorem, a small neighborhood of Δ in $M \times M$ is modeled by a neighborhood of Δ in N_Δ. This gives

$$\begin{aligned} e(M) &= \int_M \mathrm{Eul}(TM) \\ &= \int_\Delta \mathrm{Eul}(T\Delta) \\ &= \int_\Delta \mathrm{Eul}(N_\Delta) \\ &= \int_\Delta \tau_\Delta . \end{aligned}$$

We calculate the integral of τ_Δ as follows. Let $\omega_j \in Z^{p_j}(M)$ be cocycles such that $[\omega_j]$ are a basis for the vector space $H(M)$. Let $\nu_j \in Z^{m-p_j}(M)$ be forms representing the Poincare duals, i.e.

$$\int_M \omega_j \wedge \nu_k = \delta_{jk}.$$

Let $\mathrm{pr}_1, \mathrm{pr}_2 : M \times M \to M$ denote the projections to the two factors. By the Kuenneth theorem, $\mathrm{pr}_1^* \omega_j \wedge \mathrm{pr}_2^* \nu_k$ represent a basis of $H(M \times M)$.

LEMMA 14.10. *The expansion of* $[\tau_\Delta]$ *in the basis* $[\mathrm{pr}_1^* \omega_j \wedge \mathrm{pr}_2^* \nu_k]$ *reads,*

$$[\tau_\Delta] = \sum_j (-1)^{p_j} [\mathrm{pr}_1^* \omega_j \wedge \mathrm{pr}_2^* \nu_j].$$

PROOF. To find the coefficient c_{jk} of $[\mathrm{pr}_1^* \omega_j \wedge \mathrm{pr}_2^* \nu_k]$, consider the integral

$$\int_{M\times M} (\mathrm{pr}_1^* \nu_j \wedge \mathrm{pr}_2^* \omega_k) \wedge \tau_\Delta = \int_\Delta (\mathrm{pr}_1^* \nu_j \wedge \mathrm{pr}_2^* \omega_k). \tag{9}$$

The left hand side is given by

$$\begin{aligned}
& \sum_{ab} c_{ab} \int_{M\times M} (\mathrm{pr}_1^* \nu_j \wedge \mathrm{pr}_2^* \omega_k) \wedge (\mathrm{pr}_1^* \omega_a \wedge \mathrm{pr}_2^* \nu_b) \\
= & \sum_{ab} c_{ab} \, (-1)^{p_a(p_k+m-p_j)} \int_{M\times M} \mathrm{pr}_1^*(\omega_a \wedge \nu_j) \wedge \mathrm{pr}_2^*(\omega_k \wedge \nu_b) \\
= & \sum_{ab} c_{ab} \, (-1)^{p_a(p_k+m-p_j)} \int_M (\omega_a \wedge \nu_j) \int_M (\omega_k \wedge \nu_b) \\
= & \; c_{jk}(-1)^{p_j(p_k+m-p_j)}.
\end{aligned}$$

The right hand side of (9) can be written as an integral over M, using the diagonal embedding $\iota : \, M \to \Delta \subset M \times M$

$$\int_M \iota^*(\mathrm{pr}_1^* \nu_j \wedge \mathrm{pr}_2^* \omega_k) = \int_M \nu_j \wedge \omega_k = (-1)^{p_k(m-p_j)} \delta_{jk}.$$

Comparing the two results, we have found

$$c_{jk} = (-1)^{p_j \, p_k} \delta_{jk} = (-1)^{p_j} \delta_{jk},$$

□

Using this result we calculate,

$$\begin{aligned}
e(M) &= \int_\Delta \tau_\Delta \\
&= \sum_j (-1)^{p_j} \int_\Delta \mathrm{pr}_1^* \omega_j \wedge \mathrm{pr}_2^* \nu_j \\
&= \sum_j (-1)^{p_j} \int_M \omega_j \wedge \nu_j \\
&= \sum_j (-1)^{p_j} \\
&= \sum_p (-1)^p \dim H^p(M),
\end{aligned}$$

proving Theorem 14.7. The calculation just given has the following generalization. For any smooth map $F: M \to M$ from M to itself, let $\iota: \Gamma = \{x, F(x) | \, x \in M\} \hookrightarrow M \times M$ be its graph, and τ_Γ the Poincare dual.

Using our basis ω_j for $H(M)$, introduce the components of $F^*: H(M) \to H(M)$ by $F^*\omega_j = \sum_k A_{kj}\omega_k$. Thus

$$A_{kj} = \int_M (F^*\omega_j) \wedge \nu_k$$

Again, one expand $[\tau_\Gamma]$ in terms of our basis for $H(M \times M)$:

$$[\tau_\Gamma] = \sum_{jk} c_{jk} [\mathrm{pr}_1^* \, \omega_j \wedge \mathrm{pr}_2^* \, \nu_k].$$

We have

$$\int_{M \times M} (\mathrm{pr}_1^* \, \nu_j \wedge \mathrm{pr}_2^* \, \omega_k) \wedge \tau_\Gamma = \int_\Gamma \mathrm{pr}_1^* \, \nu_j \wedge \mathrm{pr}_2^* \, \omega_k.$$

The left hand side is $c_{jk}(-1)^{p_j(p_k+m-p_j)}$ as before. For the right hand side we obtain,

$$\begin{aligned} \int_\Gamma \mathrm{pr}_1^* \, \nu_j \wedge \mathrm{pr}_2^* \, \omega_k &= \int_M \iota^*(\mathrm{pr}_1^* \, \nu_j \wedge \mathrm{pr}_2^* \, \omega_k) \\ &= \int_M \nu_j \wedge F^*\omega_k \\ &= (-1)^{p_k(m-p_j)} A_{jk}. \end{aligned}$$

Comparing the two results we obtain,

$$c_{jk} = (-1)^{p_j p_k} A_{jk}.$$

If we integrate τ_Γ over the diagonal Δ, we obtain, therefore

$$\begin{aligned} \int_\Delta \tau_\Gamma &= \sum_{jk} (-1)^{p_j p_k} A_{jk} \int_\Delta \omega_j \wedge \nu_k \\ &= \sum_{jk} (-1)^{p_j p_k} A_{jk} \delta_{jk} \\ &= \sum_j (-1)^{p_j} A_{jj} \\ &= \sum_p (-1)^p \, \mathrm{tr}(F^* | \, H^p(M)). \qquad (10) \end{aligned}$$

the alternating sum of the traces of the linear maps $F^*: H^p(M) \to H^p(M)$. This integral has a nice geometric interpretation, as we shall explain in the following section.

15. Intersection numbers

Let M be a manifold, and S, S' two embedded submanifolds with $\dim S + \dim S' \geq \dim M$. One says that S, S' intersect *transversally* if for all points $x \in S \cap S'$,

$$T_xM = T_xS \oplus T_xS'.$$

EXERCISE 15.1. Suppose $S, S' \subset M$ are submanifolds of dimensions k, k' which intersect transversally. Given $x \in S \cap S'$, show that there exists a coordinate chart (U, ϕ) around x, with $\phi(x) = 0$, such that $\phi(U \cap S)$ is an open subset of $\{(x^1, \ldots, x^k, 0 \ldots, 0)\}$ and $\phi(U \cap S')$ is an open subset of $\{(0, \ldots, 0, x^{m-k'+1}, \ldots, x^m)\}$. Conclude that $S \cap S'$ is an embedded submanifold of dimension $k + k' - m$, with $T_x(S \cap S') = T_xS \cap T_xS'$.

From the exact sequence

$$0 \longrightarrow T_x(S \cap S') \longrightarrow T_xS \oplus T_xS' \longrightarrow T_xM \longrightarrow 0$$

one sees that orientations on S, S', M naturally induce an orientation on $S \cap S'$. In particular, if S, S' have complementary dimensions, $S \cap S'$ is a discrete set of points, and an orientation is given by assigning a $\pm$ sign to each element of $S \cap S'$. The plus sign appears if and only if an oriented basis of T_xS, followed by an oriented basis of T_xS', is an oriented basis for T_xM.

THEOREM 15.2. *Suppose S, S' are compact oriented submanifolds of the oriented manifold M, and that S, S' intersect transversally. Then the Poincare dual $\tau_{S \cap S'}$ of $S \cap S'$ in M can be represented by the product of Poincare duals of S, S':*

$$\tau_{S \cap S'} = \tau_S \wedge \tau_{S'}.$$

SKETCH OF PROOF. The key point is that *the pull-back of τ_S to S' is the Poincare dual of $S \cap S'$ inside S'*. (To see this, note that the restriction to $S \cap S'$ of the normal bundle N_S of S is the normal bundle of $S \cap S'$ inside S'. Hence, the Thom class $\mathrm{Th}(N_S)$ pulls back to the Thom class of $N_S|_{S \cap S'}$, since Thom forms are chratcerized by the property that theirt integral over the fibers is 1. But the Thom class of the normal bundle equals the Poincare dual for a tubular neighborhood.). We can therefore compute,

$$\int_M \alpha \wedge \tau_S \wedge \tau_{S'} = \int_{S'} \alpha \wedge \tau_S = \int_{S \cap S'} \alpha$$

for all cocycles α, which shows that $\tau_S \wedge \tau_{S'}$ represents the Poincare dual of $S \cap S'$. $\square$

In the special case that S, S' have complementary dimension and intersect transversally. the Theorem shows that the intersection number

$$i(S, S') := \int_M \tau_S \cap \tau_{S'} = \int_{S'} \tau_S = (-1)^{(m-k)(m-k')} \int_S \tau_{S'}$$

is an integer, in fact

$$i(S, S') = \int_M \tau_{S \cap S'} = \sum_{x \in S \cap S'} \epsilon(x),$$

where $\epsilon(x) = \pm 1$ according to whether the decomposition $T_xM = T_xS \oplus T_xS'$ preserves or reverses orientation. More generally, $i(S, S')$ is an integer whenever S' can be "perturbed" (i.e. homotoped) to have transversal intersection, since $i(S, S')$ is invariant under homotopies. (It is a result from differential topology that this is always possible.)

In the last section, we considered integrals $\int_\Delta \tau_\Gamma$, where $\Gamma = \{(x, F(x)\}$ is the graph of a smooth map $F : M \to M$. The intersection with the diagonal Δ is in 1-1 correspondence with the fixed points $x = F(x)$ of F. When is the intersection transversal?

LEMMA 15.3. *Δ and Γ are transversal, if and only if all fixed points x of F are* non-degenerate*, that is,*

$$\det(d_xF - I) \neq 0.$$

The sign $\epsilon(x) := \epsilon(x, x)$ at any point of intersection is equal to the sign of the determinant $\det(d_xF - I)$.

PROOF. Suppose $x = F(x)$, so that $(x, x) \in \Delta \cap \Gamma$. We have

$$T_{(x,x)}\Delta = \{(v, v)|\, v \in T_xM\}, \quad T_{(x,x)}\Gamma = \{(v, \mathrm{d}_xF(v))|\, v \in T_xM\}.$$

The map $T_xM \oplus T_xM \cong T_{(x,x)}\Delta \oplus T_{(x,x)}\Gamma \to T_{(x,x)}M = T_xM \oplus T_xM$

$$(v, w) \mapsto (v + w, v + \mathrm{d}_xF(w))$$

is described by a block matrix,

$$\begin{pmatrix} 1 & 1 \\ 1 & \mathrm{d}_xF \end{pmatrix}.$$

This has determinant,

$$\det \begin{pmatrix} 1 & 1 \\ 1 & \mathrm{d}_xF \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 1 & \mathrm{d}_xF - 1 \end{pmatrix} = \det(\mathrm{d}_xF - 1).$$

Thus, Δ and Γ are transversal if and only if $\mathrm{d}_xF - I$ is invertible, and the sign of the determinant gives $\epsilon(x, x)$. □

Putting this together with the formula (10), we obtain:

THEOREM 15.4 (Lefschetz fixed point formula). *Let $F : M \to M$ be a smooth map from a compact oriented manifold to itself, with the property that all fixed points of F are non-degenerate. For each fixed point x, let $\epsilon(x) = \mathrm{sign}(\det(d_xF - I))$. Then*

$$\sum_{x=F(x)} \epsilon(x) = \sum_p (-1)^p \,\mathrm{tr}(f^*|\, H^p(M)).$$

Note that the right hand side of this equation depends only on the smooth homotopy class of the map F.

One can obtain a similar description for the integral $\int_\Delta \tau_\Delta$. The integral is equal to $\int_\Delta \tau_\Gamma$, where Γ is the graph of a smooth map F homotopic to the identity map. To obtain such a map, pick a vector field $X \in \mathfrak{X}(M)$, and let $F^t : M \to M$ be its flow. We want that for t sufficiently small, all fixed points of F^t are non-degenerate. For small t, the fixed points of F^t are exactly the zeroes of the vector field X. Let $x \in M$ be such a zero, $X_x = 0$. Since F^t preserves x, we obtain a 1-parameter group of linear maps $\mathrm{d}_x F^t : T_x M \to T_x M$. Let $A : T_x M \to T_x M$ be the linear map $A := \frac{\partial}{\partial t}|_{t=0} \mathrm{d}_x F^t$. Then $\mathrm{d}_x F^t = \exp(tA)$. One calls A the linearization of X at x. One calls x a non-degenerate zero of X if $\det(A) \neq 0$. Let $\epsilon(x) := \mathrm{sign}(\det(A))$

We have

$$\det(\mathrm{d}_x F^t - I) = \det(tA + \ldots) = t^m \det(A) + \ldots$$

where ... denotes higher order terms in the Taylor expansion. Thus x is a non-degenerate zero for X if and only if it is a non-degenerate fixed point for F^t, for t sufficiently small. Let $t > 0$. Since $\mathrm{sign}(\det((\mathrm{d}_x F^t - I)) = \mathrm{sign}(\det(A)) = \epsilon(x)$ we obtain:

THEOREM 15.5 (Poincare-Hopf). *Let X be a vector field on a compact-oriented manifold M with non-degenerate zeroes. Then*

$$\sum_{x:\, X_x=0} \epsilon(x) = \sum_p (-1)^p \dim H^p(M).$$

where $\epsilon(x)$ is the sign of the determinant of the linearization of X at x.

One way of constructing vector fields with non-degenerate zeroes comes from Morse theory, cf. Milnor's book.

The Poincare-Hopf and Lefschetz formula can sometimes be used to obtain a lot of information on the Betti numbers, without any serious calculation.