REVIEW SHEET FOR CALCULUS 140

Some of the topics have sample problems from previous finals indicated next to the headings.

0.1 Chapters 1: Limits and continuity

Theorem 0.1.1 Sandwich Theorem*(F '96 # 20, F'97 # 12)*
If $f(x) \leq g(x) \leq h(x)$ (or $f(x) \geq g(x) \geq h(x)$) on an interval containing c and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = b$, then $\lim_{x\to c} g(x) = b$.

Definition 0.1.2 *A function $f(x)$ is defined on an interval containing c is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.*

Theorem 0.1.3 *If $f(x)$ and $g(x)$ are both continuous at c, then so are the functions $i)$ $f(x) \pm g(x)$, $ii)$ $f(x) \cdot g(x)$ and $iii)$ $f(x)/g(x)$, in case $iii)$, provided that $g(c) \neq 0$.*
If $f(x)$ is continuous at b and $g(x)$ is continuous at c, with $g(c) = b$, then $f(g(x))$ is continuous at c.

Polynomials are continuous at every point, all trigonometric functions are continuous at all points where the definition does not involve division by zero, and by the previous theorem, so all sums products, differences, quotients (excluding division by 0), and compositions of these. Once we have differentiation at our disposal, there is the following theorem about continuity.

Theorem 0.1.4 *If $f(x)$ is differentiable at c, then f is continuous at c.*

Asymptotes(S '97 # 15)
Given polynomials, $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$, $a_n \neq 0$ and $b_m \neq 0$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} i)\, a_n/b_n \text{ if } m = n, \\ ii)\, 0 \text{ if } n < m, \\ iii)\, \pm\infty \text{ if } n > m \end{cases} \qquad iv)\ \lim_{x\to\infty} \frac{f(x)}{g(x)} - (\frac{a_n}{b_{n-1}} x) = c < \infty \text{ if } m = n-1.$$

When the function $\frac{f(x)}{g(x)}$ is graphed, cases $i)$ and $ii)$ correspond to horizontal asymptotes $y = \frac{a_n}{b_n}$ and $y = 0$, respectively, and case $iv)$ corresponds to an oblique asymptote $y = \frac{a_n}{b_{n-1}} x + c$.

Theorem 0.1.5 Intermediate Value Theorem *If $f(x)$ is continuous on the interval $[a, b]$ and d is some number between $f(a)$ and $f(b)$, then there exist number c in the interval such that $f(c) = d$.*

Theorem 0.1.6 Extreme Value Theorem *If $f(x)$ is continuous on the interval $[a, b]$, then there exist numbers c, d in the interval such that $f(c) \leq f(x) \leq f(d)$ for all x in the interval.*

0.2 Chapters 2-3 and parts of Chapter 6:Differentiation

Rules for derivatives(F‘96 #14, F ‘97 #3, S ‘98 #20)

1. Product rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.
2. Quotient rule: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.
3. Chain rule: $[f(g(x))]' = f'(g(x))g'(x)$. Used in related rates problems.
4. Some derivatives: $(x^a)' = ax^{a-1}$, $(e^{ax})' = ae^{ax}$, for all real $a \neq 0$, and $(a^x)' = ln(a)a^x$, $[ln(x)]' = \frac{1}{x}$, $[log_a(x)]' = \frac{1}{ln(a)x}$ for $a > 0$.

Tangent line and linear approximation (F ‘98 # 5, F ‘98 #20, F‘99 #6, S 2000 #16)

If the function $f(x)$ is differentiable at $x = a$, the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is $y = f(a) + f'(a) \cdot (x - a)$. The function $\ell(x) = f(a) + f'(a) \cdot (x - a)$ is called the linear approximation to $f(x)$ at $x = a$. The expression $df = f'(a)dx$ is called the differential. Substituting $dx = x - a$, the differential is the linear approximation to $\Delta f = f(x) - f(a)$.

Implicit differentiation(F ‘96 #6, S ‘97 #19, F‘97 #13, S‘98 #15, S‘99 #4)

The curve described by $f(x, y) = 0$ has tangent line at the point (a, b) given by the equation $y = b + y'(a) \cdot (x - a)$, where $y'(a)$ solves the equation given by differentiating the original equation with respect to x, treating y as a function of x and using the standard rules, and finally substituting $x = a, y = b$.

For example, the curve $x^2 + xy + y^3 = 7$ has tangent line at the point $(2, 1)$ given by the equation $y = 1 - (x - 2) = 3 - x$, as we see from differentiating the equation to get $2x + y + xy' + 3y^2y' = 0$, substituting $x = 2, y = 1$ and solving for $y'(2)$, which gives $y'(2) = -1$.

Theorem 0.2.1 Mean Value Theorem *If $f(x)$ is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$, then there is a number c in the interval such that $f'(c) = [f(b) - f(a)]/(b - a)$.*

Some useful consequences of the Mean Value Theorem

Let $f(x)$ be a differentiable function defined on the interval I

1. If $f'(x) = 0$ on I, then $f(x)$ is constant.
2. If $f'(x) > 0$ on I, then $f(x)$ is increasing.
3. If $f'(x) < 0$ on I, then $f(x)$ is decreasing.(S ‘97 #14)
4. If $f''(x) > 0$ on I, then the graph of $f(x)$ is concave up.(F‘96 #18)
5. If $f''(x) < 0$ on I, then the graph of $f(x)$ is concave down.(S‘98 #7. F‘99 #7)

6. If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum. (F'98 #14, F'96 #4, Remember to check endpoints when looking for max-min on [a,b])
7. If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum. (S'98 #18)
8. If $f''(c) = 0$, and $f''(x)$ is of opposite sign for $x < c$ and $x > c$, then c is a point of inflection.(S'98 #19)

Theorem 0.2.2 l'Hôpital's rule (Chapter 6) *(S'98 #2, F'98 # 1,2,4, F'99 #16)*
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or both limits equal* ∞*, and* $g(x) \neq 0$ *on an interval containing* a*, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Some examples:

1. $\lim_{x\to 0} sin(x)/x = \lim_{x\to 0} cos(x)/1 = 1.$
2. $$\lim_{x\to 0} x^a ln(x) = \lim_{x\to 0} ln(x)/x^{-a} = \lim_{x\to 0}[1/x]/[(-a)x^{-a-1}] = \lim_{x\to 0} \frac{1}{(-a)} x^a = 0 \text{ for all } a > 0.$$
3. $$\lim_{x\to\infty} \frac{x^n}{e^x} = \text{ differentiating } n\text{-times } \lim_{x\to\infty} \frac{n!}{e^x} - 0 \text{ for all positive integers n.}$$
4. $\lim_{x\to 0}(1 + ax)^{\frac{1}{x}} = \lim_{x\to\infty}(1 + \frac{a}{x})^x = e^a,$ for all a.

The third example is proved using the following useful fact If $\lim_{x\to a} ln(f(x)) = L$, then $\lim_{x\to a} f(x) = e^L$.

Exponential growth and decay: (F '97 # 19, S '98 # 3 (growth), F '99 #14)

$$A(t) = A_0 e^{kt}, \quad \text{where } A_0 \text{ is the initial amount.}$$

For radioactive decay, $k = \frac{-ln(2)}{T}$ where T is the half-life.

Derivative of the inverse function(F'96 #23, F '97 #16, S'97 #6, S '99 #15)

If $f(x)$ is one-to-one on the interval I, then there is an inverse function on the image of the interval, $f(I)$. If $f(x)$ is differentiable at the point a in I and $f'(a) \neq 0$, then the inverse function $f^{-1}(x)$ is differentiable and

$$[f^{-1}(x)]'|_{x=f(a)} = 1/[f'(a)].$$

0.3 Chapters 4-7 : Integration

Riemann sums: (F'96 #11)

If $f(x)$ is continuous on the interval $[0,1]$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}f(\frac{i}{n}) = \int_0^1 f(x)dx.$$

Initial value problems (S'98 #16)

The initial value problem $y' = f(x)$, $y(a) = b$ is solved by first finding the indefinite integral (set of antiderivatives) of $f(x)$ and then choosing the "free constant" so that the value of the antiderivative at $x = a$ is b.

Example: $y' = 2x^3 + x$, $y(1) = 2$. The indefinite integral of $2x^3 + x$ is $\frac{1}{2}(x^4 + x^2) + c$. Setting $x = 1$, we see that $1 + c = 2$, so $c = 1$. The solution is $y = \frac{1}{2}(x^4 + x^2) + 1$.

Integration by substitution or reading the chain rule backwards(Many examples, S '97 #3, F'97 #10, S'98 #6, F'98 #19, S'99, #11, F'99 #10, 11)

$$\int f(u(x))u'(x)dx = \int f(u)du, \text{ or as a definite integral } \int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du.$$

Integration by parts or reading the product rule backwards (F'96 #13, S'97 #1)

$$\int udv = uv - \int vdu, \text{ as a definite integral } \int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.$$

Average value of $f(x)$

$$min_{a<x<b}f(x) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq max_{a<x<b}f(x).$$

Theorem 0.3.1 Fundamental Theorem of Calculus *(F'97 #6,S'97 #11)*

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Note: To differentiate $\int_a^{u(x)} f(t)dt$ with respect to x, when the upper limit is a function $u(x)$, use the chain rule.

Area between curves(F'97#15, F'99 #9)

If $f(x)$ and $g(x)$ are continuous on $[a,b]$ and $f(x) \geq g(x)$ on that interval, then the area of the region bounded by the graphs $y = f(x)$, $y = g(x)$ and the vertical lines $x = a$, $x = b$ is given by the definite integral $\int_a^b [f(x) - g(x)]dx$.

If the region is between curves $x = h(y)$ and $x = k(y)$ which are graphs over the x-axis with $h(y) \leq k(y)$ on the interval $c \leq y \leq d$ then the area of the region bounded by the graphs and the horizontal lines $y = c$ and $y = d$ is $\int_c^d [k(y) - h(y)]dy$.

A region may satisfy both conditions. For example the region between the curves $y = 2x$ and $y = x^2$ lies over the interval $0 \leq x \leq 2$, but it is also the region between $x = \frac{y}{2}$ and $x = \sqrt{y}$ over the interval $0 \leq y \leq 4$. The area is calculated either by

$$\int_0^2 [2x - x^2]dx = \frac{4}{3} \text{ or } \int_0^4 [\sqrt{y} - \frac{y}{2}]dy = \frac{4}{3}.$$

Volumes by slicing(F'96 #16,19, F'97 B2, F'98 #9)

The volume of known cross-sectional area $A(x)$ from $x = a$ to $x = b$ is given by the integral $\int_a^b A(x)dx$. This gives the **Washer formula** for the volume of a solid of revolution given by rotating the region between the curves $y = f(x)$ and $y = g(x)$ over the interval $[a, b]$ around the x-axis, (where $f(x) \geq g(x)$ on $[a, b]$) :

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx.$$

If the region is also described as being between the curves $x = k(y)$ and $x = h(y)$ over the interval $c \leq y \leq d$, but we still rotate it around the x-axis, then the volume is given by the method of **Cylindrical shells**:

$$V = \int_c^d 2\pi y[k(y) - h(y)]dy.$$

Note that the formula in the book on page 389 uses $f(y)$ instead of $k(y) - h(y)$. These both mean the height of the cylindrical shell. In the washer formula the generating segment is perpendicular to the axis of rotation, whereas, in the cylindrical shell formula the generating segment is parallel to the axis. See page 391 in the book.

Partial fractions (S '97 #4, F'97#9,10 F'98 #6)

To integrate a rational function $f(x)/g(x)$, first make sure the expression is in reduced form with $deg(f) < deg(g)$ (dividing if necessary). Then factor the denominator $g(x)$ into a product of linear factors $x - r$ corresponding to real roots and quadratic factors $x^2 + bx + c$ corresponding to compex roots. Then expand the expression in partial fractions as a sum of terms $\frac{C}{(x-r)^m}$, where m runs from 1 to the highest power of $x - r$ in the factorization of $g(x)$ and terms $\frac{Ax+B}{(x^2+bx+c)^k}$, where k is the highest power of $x^2 + bx + c$ in the factorization of $g(x)$. In most of the examples, the highest power for both cases is 1.

Remember that when $x - r$ occurs just once as a factor in $g(x)$, then the coefficient C in $\frac{C}{(x-r)}$ is easily calculated by cancelling the term $x - r$ in $g(x)$ and substituting $x = r$ in the remaining terms of $f(x)$ and $g(x)$.

For example: $1/[(x - 1)(x - 2)(x - 3)] = \frac{a}{(x-1)} + \frac{b}{(x-2)} + \frac{c}{(x-3)}$. So $a = 1/[(-1)(-2)]$, $b = 1/[(1)(-1)]$ and $c = 1/[(2)(1)]$.

$$\begin{aligned}
\int dx/[(x-1)(x-2)(x-3)] &= \int \frac{1/2}{(x-1)}dx + \int \frac{-1}{(x-2)}dx + \int \frac{1/2}{(x-3)}dx \\
&= \frac{1}{2}\ln|x-1| - ln|x-2| + \frac{1}{2}ln|x-3| \\
&= ln(\sqrt{|(x-1)(x-3)}/|x-2|).
\end{aligned}$$

Improper integrals (S'97 #9, F '98 # 8, S '98 # 5, S'99 #24)

If $f(x)$ is continuous on the half open interval $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$ then the $\int_a^b f(x)dx$ is called an **improper integral** and it is defined as a limit:

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

Similarly, if $f(x)$ is continuous on the half open interval $[a,b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

Integrals over an infinite interval are also improper integrals, and are defined as limits

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx, \quad \int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

Examples:

$$\begin{aligned}
\int_a^\infty \frac{dx}{x^k} &= \lim_{b\to\infty} \int_a^b \frac{dx}{x^k} \\
&= \begin{cases} \lim_{b\to\infty} \frac{1}{1-k} x^{1-k}|_a^b = \lim_{b\to\infty} \frac{1}{1-k} b^{1-k} - \frac{1}{1-k} a^{1-k} \text{ for } k \neq 1 \\ \lim_{b\to\infty} lnb - lna \text{ for } k = 1. \end{cases} \\
&= \begin{cases} a^{1-k}/(k-1) \text{ for } k > 1 \\ \infty \text{ for } k \leq 1. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\int_0^b \frac{dx}{x^k} &= \lim_{a\to 0^+} \int_a^b \frac{dx}{x^k} \\
&= \begin{cases} \lim_{a\to 0^+} \frac{1}{1-k} b^{1-k} - \frac{1}{1-k} a^{1-k} \text{ for } k \neq 1 \\ \lim_{a\to 0^+} lnb - lna \text{ for } k = 1. \end{cases} \\
&= \begin{cases} b^{1-k}/(1-k) \text{ for } k < 1 \\ \infty \text{ for } k \geq 1 \end{cases}
\end{aligned}$$

Another example (using the calculation from the partial fractions example given above)

$$\begin{aligned}
\int_4^\infty dx/[(x-1)(x-2)(x-3)] &= \\
\lim_{b\to\infty} ln(\sqrt{|(b-1)(b-3)}/|b-2|) &- ln(\sqrt{|(4-1)(4-3)}/|4-2|) = ln(\sqrt{3}/2),
\end{aligned}$$

because $\lim_{b\to\infty}(\sqrt{|(b-1)(b-3)}/|b-2|) = 1$ and $ln(1) = 0$.

However, if we consider the improper integral $\int_3^\infty dx/[(x-1)(x-2)(x-3)]$, then we need to pick some midpoint where there is no discontinuity, for example, $x = 4$ and consider limits at both endpoints

$$\int_3^\infty dx/[(x-1)(x-2)(x-3)] = \lim_{a\to 3^+} \int_a^4 dx/[(x-1)(x-2)(x-3)] + \lim_{b\to\infty} \int_4^b dx/[(x-1)(x-2)(x-3)].$$

We just calculated he second limit and it is finite, but the first limit

$$\lim_{a\to 3^+} \int_a^4 dx/[(x-1)(x-2)(x-3)] = \lim_{a\to 3^+} [ln(\sqrt{3}/2) - ln(\sqrt{|(a-1)(a-3)}/|a-2|)]$$

is infinite, so the improper integral $\int_3^\infty dx/[(x-1)(x-2)(x-3)]$ diverges.