Fluctuation-Dissipation Equation and Incompressible Navier-Stokes Equations

Jeremy Quastel* and Horng-Tzer Yau†

February 26, 1998

Abstract

We review some recent results concerning the hydrodynamical limits of lattice gases, in particular, lattice gases on the cubic lattice $\mathbb{Z}^d$, $d = 3$, with the incompressible Navier-Stokes equations as the hydrodynamical limit. We shall state precisely the law of large numbers theorem stating that for initial distributions corresponding to arbitrary macroscopic L^2 initial data the distributions of the evolving empirical momentum densities are supported entirely on global weak solutions of the incompressible Navier-Stokes equations. Furthermore the rate function for the large deviation will be described. The viscosity will be characterized by the Green-Kubo formula and variational principles. The key analytic inputs are a method solving the fluctuation-dissipation equation of the lattice gases and multiscale estimates based on the logarithmic Sobolev inequality. A precise formulation of the fluctuation-dissipation equation will be sketched. Some discussions of its relation to the Green-Kubo formula and the variational formulas will also be sketched.

1 Introduction

Although the fundamental equations for a fluid such as the Euler equations or the Navier-Stokes equations have been known and widely used for centuries, basic issues concerning these equations remain unanswered. The equations are on the macroscopic scale and are derived from a continuum formulation of conservation of mass and momentum. But one can ask the deeper question of how to obtain these equations directly from microscopic models, in particular from many body Hamiltonian systems, which in principle governs all classical phenomenon. Then one can ask for the microscopic origin of the viscosity, the nonlinearity, and the next order corrections to the macroscopic equations. Furthermore, intrinsic microscopic questions such as fluctuations, time-dependent correlation functions or the behavior of tagged particles are naturally formulated on the microscopic level.

At the present time we are very far from a rigorous derivation of fluid equations from Hamiltonian mechanics. This is because the ergodic properties of many body Hamiltonian systems are very poorly understood. The best that has been done rigorously is to derive the Euler equations from Hamiltonian systems under some weak ergodicity assumptions [17]. Another approach is to derive the Euler and the Navier-Stokes equations from the Boltzmann equation. This approach has its origin from the work of Chapman-Enskog-Hilbert and many rigorous results have been obtained in the seventies and eighties, in particular, [2, 3, 5, 1]. The Boltzmann equation is not really a microscopic model; it should itself be derived as a scaling limit of a more basic model. The fundamental result of O. Lanford [15] provides a derivation of the Boltzmann equation from the Hamiltonian dynamics for short time in a low density limit. So the fluid dynamic limit of the Boltzmann equation will necessarily be that of an ideal gas. Furthermore, the Chapman-Enskog expansion can be rigorously justified only in the smooth regions of the macroscopic equations. Though there are partial results to go beyond the smooth regions, the program [1] of deriving the weak (Leray) solutions of the incompressible Navier-Stokes equations from the DiPerna-Lions solutions of the Boltzmann equation remains incomplete.

*Department of Mathematics, University of California, Davis, CA 95616. e-mail: quastel@math.ucdavis.edu. Partially supported by U. S. National Science Foundation grant DMS-9504791.

†Courant Institute of Mathematical Sciences, New York University, New York, NY, 10012, USA. email: yau@math.nyu.edu. Partially supported by U. S. National Science Foundation grants DMS-9403462, 9703752, and David and Lucile Packard Foundation Fellowship.

An alternate approach is to substitute at the microscopic level various simplified and regularized versions for the Hamiltonian systems. One class of simplified models are the lattice gases or cellular automata, in which the particles are confined to a lattice and given deterministic motions and collision rules in order to make the system as close as possible to the Hamiltonian system. Such models have been used extensively for simulations of fluids. However, the discrete nature does not really simplify the analysis. The lattice structure tends to impose spurious conservation laws on the dynamics, and the problem of ergodicity remains.

One can consider instead the stochastic versions of the lattice gases. The stochasticity regularizes the model, causes the degenerate states, which are a source of such difficulty in the deterministic models, to become unstable. This approach in particular was initiated by Spitzer, Dobrushin, Lebowitz and developed by Presutti and Spohn. Notice that up to the end of the eighties the approach through the Boltzmann equation was far advanced in the derivation of the Euler or the Navier-Stokes equations when compared with the lattice gases, mostly because the Boltzmann equation is a macroscopic equation and its mathematical analysis uses methods widely available in the study of partial differential equations. Lattice gases, on the other hand, are infinite dimensional systems and analytical tools have not been fully developed. In fact, most approaches relied on perturbation methods or adapted partial differential equation techniques to lattice gases until the entropy method initiated by Varadhan [10, 22] in the eighties. Though this approach was originally designed for reversible systems with nonlinear diffusion equations as the macroscopic equations, later development has successfully extended it to nonreversible systems with incompressible Navier-Stokes equations as macroscopic equations. In this review, I shall highlight some results related to the derivation of the incompressible Navier-Stokes equations. Interested readers should consult the book of Spohn [21] for an extensive review.

To derive the incompressible Navier-Stokes equations, it helps first to understand the scaling limit under which we shall obtain our result. The special scaling limit needed for the incompressible Navier-Stokes equations is called the incompressible limit, which is the usual diffusive limit but with the velocity rescaled by ϵ^{-1}. This means the velocity is small compared with the velocity of the sound [6, 9]. Our second remark concerns the dimension. We shall restrict ourselves to dimension $d = 3$. The restriction $d \leq 3$ is for technical reasons because the regularity of the incompressible Navier-Stokes equations for $d > 3$ available nowadays is too weak for our purpose; *the restriction $d \geq 3$, however, is intrinsic.* Since the macroscopic velocity is defined through the law of large numbers in statistical physics, it inherits a small fluctuation from the central limit theorem, which is of order $\epsilon^{d/2}$. Since we have to blow up the velocity by a factor of order ϵ^{-1} in the incompressible limit, this fluctuation becomes of order one or bigger for dimension $d \leq 2$ and hence the blowed-up macroscopic velocity is not well defined in this limit. Notice that this argument applies to any dynamics including the Hamiltonian dynamics. The trouble with dimension $d \leq 2$ also manifests in the Green-Kubo formula, which expresses the viscosity as a time integration of the current-current correlation functions. A key question concerning the Green-Kubo formula is whether the time integration converges. This is not a purely mathematical question. Indeed, one expects that the Green-Kubo formula diverges in dimension $d = 2$ and the time scaling for two dimensional lattice gases is no longer diffusive. We are thus forced to conclude that the two dimensional incompressible Navier-Stokes equations cannot be obtained as the incompressible limit of *any* microscopic dynamics.

The current-current correlation functions are difficult to control for lattice gas dynamics. A proper approach is through the fluctuation-dissipation equation, to be explained in certain detail later on. Notice that the Green-Kubo formula or the fluctuation-dissipation equation is an equilibrium theory. So a link to nonequilibrium is needed. For this purpose, we need to combine the entropy method of [22] with the logarithmic Sobolev inequality, see [8, 19] for details. Up to this point, all the invariant measures are product, namely, the temperature is infinite. To bring into the temperature, one needs to replace the invariant measures by Gibbs states at finite temperature. The results in this direction are much more limited, partly due to the lack of analytic tools for phase transitions. We shall only report briefly some partial results in the last section.

All results described in this review are from joint works with C. Chang, R. Esposito, C. Landim, R. Marra, S. Olla, H. Spohn, S. Sethuraman, and S. Varadhan. We would like to take this opportunity to thank all of them. Detailed reference will be given in the last section.

2 A lattice gas model for the incompressible Navier-Stokes equations

Our goal in this section is to review a model [8] with the incompressible Navier-Stokes equations as the macroscopic limit. Let $\Lambda_L \in \mathbb{Z}^d$, $d = 3$ be the cubic lattice $\{-L, ..., L\}^d$ with periodic boundary conditions. We put $L = \varepsilon^{-1}$ and denote by $e_j,\ j = 1, ..., d$ the unit vectors in the positive coordinate directions. Let $\mathcal{V} \subset \boldsymbol{R}^d$ be a finite set representing the possible velocities. On each site of the lattice at most one particle for each velocity is allowed. A configuration of particles on the lattice is denoted by $\eta = \{\eta_x,\ x \in \Lambda_L\}$ where $\eta_x = \{\eta(x, v),\ v \in \mathcal{V}\}$ and $\eta(x, v) \in \{0, 1\},\ x \in \mathbb{Z}^d,\ v \in \mathcal{V}$ is the number of particles with velocity v at x. The set of particle configurations is $X = \left(\{0,1\}^{\mathcal{V}}\right)^{\Lambda_L}$. The dynamics consists of two parts: Asymmetric random walk with exclusion among particles of the same velocity and binary collisions between particles of different velocities.

We first describe the random walk part of the dynamics. Particles of velocity v perform a continuous time asymmetric random walk with simple exclusion. A particle at x waits a random time, exponentially distributed with mean one, then chooses a nearby site $x + y$ with probability $p(y, v)$ and jumps there as long as the site is not occupied by another particle of the same velocity. If there is a particle of the same velocity, the jump is suppressed and the particle waits for a new exponential time. All particles are doing this simultaneously, and since time is continuous ties do not occur. The jump law $p(y, v)$ should be finite range, irreducible and have mean velocity v:

$$\sum_y y p(y, v) = v.$$

For the sake of concreteness, we will take in this paper the jump law to be nearest neighbor with

$$p(e, v) = \gamma + e \cdot v/2$$

for each standard basis vector e. We assume here that γ is large enough that all the $p(e, v)$ are non-negative. The generator of the random walk part of the dynamics is given by $\mathcal{L}^{ex}$ which acts on functions f on the configuration space X by

$$\mathcal{L}^{ex} f(\eta) = \sum_{x, v\in\mathcal{V}, |e|=1} \tilde{p}(x, e, v; \eta) \left(\nabla^v_{x,e} f\right)(\eta) \tag{2.1}$$

where e denotes unit vectors along coordinate directions,

$$\tilde{p}(x, e, v; \eta) = \eta(x, v)(1 - \eta(x + e, v))p(e, v)$$

is the jump rate from x to y, and $\nabla^v_{y,e}$ is the lattice gradient given by

$$\left(\nabla^v_{y,e} f\right)(\eta) = f(\eta^v_{y,e}) - f(\eta)$$

where

$$\eta^v_{y,e}(z, w) = \begin{cases} \eta(z, w) & \text{if } v \neq w \text{ or } z \neq x, x + e; \\ \eta(y, v) & \text{if } v = w \text{ and } z = y + e; \\ \eta(y + e, v) & \text{if } v = w \text{ and } z = y. \end{cases}$$

The sum in (2.1) is over all $x \in \Lambda_L$. Note that we will also use the symbol $\nabla\phi$ to denote the usual gradient of a function $\phi : T^d \to \boldsymbol{R}$. The lattice gradient will always be written with sub and super-indices to distinguish it from the usual gradient of a smooth function.

The collision part of the dynamics is described as follows. Particles of velocities v and w at the same site collide and produce two particles of velocities v' and w' at that site at rate $\tilde{r}(q)$ where q denotes the collision $(v, w) \mapsto (v', w')$. The set of such collisions will be denoted $\mathcal{Q}$. The generator of the collisions is given by

$$\mathcal{L}^c f(\eta) = \sum_{q\in\mathcal{Q},\ x} r(x, q; \eta) \left(Q^q_x f\right)(\eta)$$

where if $q = (v, w) \mapsto (v', w')$,

$$r(x, q; \eta) = \eta(x, v)\eta(x, w)(1 - \eta(x, v'))(1 - \eta(x, w'))\tilde{r}(q)$$

and Q^q_x is the collision operator given by

$$\left(Q^q_x f\right)(\eta) = f(Q^q_x \eta) - f(\eta)$$

where

$$Q_x^q\eta(y,u) = \begin{cases} \eta(y,u) & \text{if } y \neq x \text{ or } u \neq v, w, v', w' \\ 0 & \text{if } y = x \text{ and } u = v, w \\ 1 & \text{if } y = x \text{ and } u = v', w' \end{cases}$$

We now give an example of a set of velocities $\mathcal{V}$ satisfying the assumptions. Let σ be an element of $\mathcal{T}$, the permutation group of $\{1,2,3\}$. If $v \in \boldsymbol{R}^3$ denote $\sigma v = (v_{\sigma 1}, v_{\sigma 2}, v_{\sigma 3})$. Let

$$\mathcal{V} = \{v : \sigma v = (\pm 1, \pm 1, \pm \omega) \text{ for some } \sigma \in \mathcal{T}\}$$

where ω is the positive solution of

$$\omega^4 - 6\omega^2 - 1 = 0.$$

The rate of collisions $q = (v, w) \mapsto (v', w')$ is given by

$$\tilde{r}(q) = \begin{cases} 1 & \text{if } v + w = v' + w'; \\ 0 & \text{otherwise.} \end{cases}$$

The full dynamics consist of the random walk part and the collision part running at the same time, and hence the generator is given by $\mathcal{L} = \mathcal{L}^{ex} + \mathcal{L}^c$. This is called Model II in [8].

The invariant measures can be described as follows. Let I_0 denote the local density and I_α the local velocity defined by

$$I_0(\eta_x) := \sum_{v \in \mathcal{V}} \eta(x,v), \qquad I_\alpha(\eta_x) := \sum_{v \in \mathcal{V}} v_\alpha \eta(x,v), \quad \alpha = 1, ..., d, \tag{2.2}$$

where v_α is the αth coordinate of the vector v, and η_x denotes the configuration at site x. Then the invariant measures are given by the convex hull of the Bernoulli measures

$$\mu_\lambda = Z_{L,\lambda}^{-1} \exp\left\{ \sum_{x \in \Lambda_L} \sum_{\alpha=0}^{d} \lambda_\alpha I_\alpha(\eta_x) \right\};$$

here λ is a parameter called the chemical potential. The particular case when $\lambda_0 = r$ for some real number r, and $\lambda_\alpha = 0$ for $\alpha = 1, \ldots, d$ will be denoted μ_r and will be used throughout as a reference measure. Any distribution can be written as $f d\mu_r$ and the meaning of the generator is that if $f_0 d\mu_r$ is the initial distribution of our process and $f(t)d\mu_r$ is the distribution at time t then $f(t)$ is obtained from f_0 by solving the evolution equation

$$\frac{\partial f}{\partial t} = \mathcal{L}^* f, \quad f(0) = f_0.$$

where $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$ with respect to the reference measure μ_r.

3 The incompressible limit

Recall the INS:

$$\{$$

It is invariant under the scaling:

$$x \to \varepsilon x, \quad t \to \varepsilon^2 t, \quad u \to \varepsilon^{-1} u, \quad p \to \varepsilon^2 p.$$

This determines the scaling, called the incompressible scaling. In other words, we consider velocities of order ε, varying on spacial scales of order ε^{-1}, and on time scales of order ε^{-2}. This is the regime in which velocities are very small compared to the speed of sound and is emphasized in [6, 9] as the correct scaling for a derivation of incompressible Navier-Stokes equation from particle systems. Corresponding to this rescaling we speed up the microscopic time by a factor of ε^{-2} to correspond to the macroscopic time scale,

$$\mathcal{L} \mapsto \varepsilon^{-2} \mathcal{L}.$$

and we rescaled the empirical momentum by defining for each configuration η the rescaled empirical momentum densities as the following measures on the torus T^d,

$$\nu_\alpha(dy) := \varepsilon^{d-1} \sum_{x \in \Lambda_L} \delta_{\varepsilon x} I_\alpha(\eta_x), \qquad \alpha = 1, ..., d. \tag{3.1}$$

Here δ_y denotes the Dirac delta measure at $y \in T^d$.

The initial state of the system is given by the local equilibrium distribution,

$$d\mu_\lambda^\varepsilon = Z_{\lambda(\cdot),\varepsilon}^{-1} \exp\left\{ \sum_{x\in\Lambda_L} \left(\lambda_0(\varepsilon x) I_0(\eta_x) + \sum_{\alpha=1}^{d} \lambda_\alpha(\varepsilon x) I_\alpha(\eta_x) \right) \right\} \tag{3.2}$$

In order to be in the incompressible regions, the chemical potential takes the form

$$\lambda_0(\varepsilon x) = r + \varepsilon^2 \lambda_0^{(2)}(\varepsilon x), \qquad \lambda_\alpha(\varepsilon x) = \varepsilon \lambda_\alpha^{(1)}(\varepsilon x).$$

where $\lambda_0^{(2)}$ and $\lambda_\alpha^{(1)}$ are given smooth functions on the torus. Corresponding to this initial distribution and the given dynamics we have a measure Q_ε on $D([0,T];X)$, the set of right continuous trajectories with left limits with values in the configuration space X, equipped with the Skorohod topology. On the torus in $d = 3$, the Sobolev space H_2 of functions with two square integrable derivatives is embedded in C, the space of continuous functions. Therefore the space of measures $\mathcal{M}$ is embedded in the dual H_{-2} of H_2 through the L^2 pairing. The Sobolev spaces are more convenient for our purposes and therefore we will consider the empirical momentum densities as elements of H_{-2}. The probability measure Q_ε on $D([0,T];X)$ is then carried by (3.1) to a probability measure P_ε on $L^2(0,T;H_{-2})$. In other words, P_ε is the distribution of the trajectories $\nu_\alpha(t,d\theta)$ considered as elements of $L^2(0,T;H_{-2})$. The elements here have of course d components and we will always mean the d-fold product of L^2. Let $u^0 \in L^2$ be given and suppose that the initial distributions $\lambda^{(1)}(x)$ are chosen so that for all smooth test functions ψ and each $\delta > 0$, there is a u^0 such that

$$\lim_{\varepsilon\to 0} \mu_\lambda^\varepsilon \left(\left| \int \psi d\nu - \int \psi u^0 dx \right| \geq \delta \right) = 0.$$

From the law of large number for the product measure $d\mu_\lambda^\varepsilon$, one can compute u^0 explicitly from $\lambda^{(1)}$ and vise versa. We omit the simple formula. The main result of law of large numbers type is

Theorem 3.1. *Let P_ε be the distributions of the empirical momentum densities (3.1). Then there is a viscosity D such that P_ε are tight as a set of probability measures on the space $L^2(0,T;H_{-2})$ with the weak topology, and any weak limit P is supported entirely on weak solutions of the INS equations satisfying the energy inequality.*

Next we consider the large deviations from this scaling limit. There are in fact two different large deviation rates: Off the hyperplane of divergence free measures, the decay rate is at least $\exp\{-\varepsilon^{-d+1}\}$, while on that hyperplane the decay rate is $\exp\{-\varepsilon^{-d+2}\}$. This corresponds to the fact that there is an initial layer in which the gradient part of the velocity field is removed. At the present time we are unable to pin down the exact rate of this faster process. On the other hand, for the second, slower part we have precise asymptotics.

Recall that P_ε has initial distribution given by (3.2) , corresponding to initial data u_0 for the INS. To observe the slower decay of the violation of the momentum conservation equation one must project out the divergence part of the velocity field. Define H_1 the Hilbert space with the inner product

$$\langle \phi, \psi \rangle = \int (\nabla\phi, D\nabla\psi) dx.$$

Since D is strictly positive definite this definition of H_1 is equivalent to the standard one. But the precise form will be useful. Let

$$V = \text{closure of } \left\{ \psi \in C^\infty \;\middle|\; \nabla\cdot\psi = 0 \right\} \text{ in } H_1;$$

and

$$H = \text{closure of } \left\{ \psi \in C^\infty \;\middle|\; \nabla\cdot\psi = 0 \right\} \text{ in } L^2.$$

Denote by $\mathbf{P}$ the orthogonal projection $\mathbf{P} : L^2 \to H$. $\mathbf{P}$ is known as the Leray projector. The Leray projector extends to an orthogonal projection $\mathbf{P}$ from H_{-2} to the closure of H in H_{-2}. Let $\hat{P}_\varepsilon$ on $L^2(0,T;H_{-2})$ with the weak topology be the distributions of the divergence free part $u = \mathbf{P}\nu$ of the empirical momentum densities. Let V' be the dual space of V and define

$$\mathcal{I}(u,f) = \frac{1}{4\chi} \left\| \frac{\partial u}{\partial t} + c\mathbf{P}(u\cdot\nabla u) - \nabla D \nabla u - f \right\|^2_{L^2(0,T;V')}.$$

The main result for the large deviation is given by the following theorem.

Theorem 3.2. *$\tilde{P}_\varepsilon$ have the large deviations property with rate function $\mathcal{I}$ and rate ε^{2-d}. In particular, for each closed set $\mathcal{U}$ and open set $\mathcal{O}$,*

$$\limsup_{\varepsilon\to 0} \varepsilon^{d-2} \log \tilde{P}_\varepsilon(\mathcal{U}) \le - \inf_{u(\cdot,\cdot)\in\mathcal{U}} \mathcal{I}(u(\cdot,\cdot))$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon^{d-2} \log \tilde{P}_\varepsilon(\mathcal{O}) \ge - \inf_{u(\cdot,\cdot)\in\mathcal{O}} \mathcal{I}(u(\cdot,\cdot)).$$

The rate function described here contains only the dynamical part of the full rate function and is, strictly speaking, not rigorously defined. See [19] for detail.

4 Green-Kubo formula and the fluctuation-dissipation equation

It remains to characterize the viscosity matrix $D^{i,\ell}_{j,k}$, formally given by the Green-Kubo formula. Define the currents w^i_j by

$$\mathcal{L}[I_i(\eta_x)] = \sum_{j=1}^{d} \nabla_j^- w^i_{x,j} \tag{4.1}$$

where I is the full set of conserved quantities defined in (2.2). ∇_j^- is the lattice gradient acting on functions on the lattice by $\nabla_j^- f(n) = f(n) - f(n-e_j)$, and $w^i_{x,j} = \tau_x w^i_j$. Let $\mathcal{G}$ be the space of local functions satisfying $E^{\mu_\lambda}[g] = 0$ and $E^{\mu_\lambda}[g;\bar{I}_\alpha] =: E^{\mu_\lambda}[g\bar{I}_\alpha] - E^{\mu_\lambda}[g]E^{\mu_\lambda}[\bar{I}_\alpha] = 0$ for $\alpha = 0,\ldots,d$. Denote by σ^i_j the parts of the current orthogonal to the constants of motion, i.e.,

$$\sigma^i_j = w^i_j - \sum_{\alpha=0}^{d} c^i_{j,\alpha} I_\alpha$$

where the coefficients $c^i_{j,\alpha}$ are chosen so that $\sigma^i_j \in \mathcal{G}$. With this notations, the Green-Kubo formula is given by

$$D^{i,\ell}_{j,k} = \gamma\delta_{i\ell}\delta_{jk} + \chi^{-1}\int_0^\infty dt \sum_x \Big\langle \sigma^i_j, e^{\mathcal{L}t}\tau_x\sigma^\ell_k \Big\rangle.$$

Here τ_x is the shift by x on the lattice, $\chi = \Big\langle \Big(I_\alpha - \Big\langle I_\alpha \Big\rangle\Big)^2 \Big\rangle$ is the susceptibility and the expectations $\Big\langle \cdot \Big\rangle$ are with respect to the reference invariant measure for our process μ_r. Although the Green-Kubo characterize the viscosity in a closed formula, it is however of limited application. The first reason is the formula involving the semigroup, which in general is very complicated. Moreover, the time integration in the formula may not converge. Indeed, one expects that the Green-Kubo formula diverges in dimension $d = 2$ and the time scaling for two dimensional lattice gases is no longer diffusive. We now describe the Green-Kubo formula in another formulation through the fluctuation-dissipation equation.

Roughly speaking, the fluctuation-dissipation equation is an equation for u solving

$$w = \mathcal{L}u + D\nabla I$$

for some constant matrix D, identified to be the viscosity; here w is the current defined in (4.1). Intuitively, $\mathcal{L}u$ represents rapid fluctuation and the equation (4) describes a decomposition of the current into fluctuation and gradient of the density field, representing the dissipation. For this reason, it is called the fluctuation–dissipation equation. As it stands, (4) is just a formal relation. We need to specify the precise meaning of this equation.

For local functions f and g of mean zero let

$$\ll f,g \gg_0 = \Big\langle f, \sum_{x\in\mathbb{Z}^d} \tau_x g \Big\rangle = \Big\langle g, \sum_{x\in\mathbb{Z}^d} \tau_x f \Big\rangle$$

where the expectation is w.r.t. the invariant measure μ_r. Up to a null set, which can be factored out, this defines an inner product. We can then take completions of this space and obtain a Hilbert space, call $\mathcal{H}_0$. The generator $\mathcal{L}$ extends naturally to an operator on $\mathcal{H}_0$. We can decompose $\mathcal{L}$ into a symmetric part $\mathcal{L}_s$ and an antisymmetic part $\mathcal{A}$. Define $\mathcal{H}_1$ as the space with the inner product

$$\ll f, g \gg_1 = \left\langle f, (-\mathcal{L}_s) \sum_{x \in \mathbb{Z}^d} \tau_x g \right\rangle$$

Explicitly,

$$\ll f, f \gg_1 = \frac{1}{2} E^{\mu_r} \left[\gamma \sum_{\substack{x, e > 0, \\ v \in \mathcal{V}}} \left(\nabla_{x,e}^v f\right)^2 + \sum_{\substack{x, \\ q \in \mathcal{Q}}} r(x, q; \eta) \left(Q_x^q f\right)^2 \right].$$

These spaces were introduced by [21]. Now from $\mathcal{H}_1$ one can defined naturally its dual $\mathcal{H}_{-1}$ through the pairing in $\mathcal{H}_0$. We emphasize that all these spaces depend only on the symmetric part of the operator, which is a much simpler operator compared with the full operator $\mathcal{L}$. Our major results are

Theorem 4.1. *Suppose the dimension $d \geq 3$. Then there is an matrix D such that*

$$\inf_{u \in \mathcal{G}} \|w - D\nabla I - \mathcal{L}u\|_{-1} = 0$$

Furthermore, the matrix D is given by the Green-Kubo formula and it satisfies the following variaitonal formulas. For $a = (a_j^\alpha)$ and $b = (b_j^\alpha)$, $j, \alpha = 1, \ldots, d$, let

$$(a, Db) = \sum_{\alpha, i, \beta, j = 1}^{d} D_{i,j}^{\alpha,\beta} a_i^\alpha b_j^\beta.$$

Then

$$(a, Da) = \frac{\gamma}{\chi} \inf_{F \in \mathcal{G}} \langle\langle \sigma \cdot a + \mathcal{L}F, \sigma \cdot a + \mathcal{L}F \rangle\rangle,$$

and

$$\left(a, D^{-1}a\right) = \frac{1}{\gamma\chi} \inf_{F \in \mathcal{G}} \langle\langle \nabla I \cdot a + \mathcal{L}F, \nabla I \cdot a + \mathcal{L}F \rangle\rangle.$$

5 Further results and concluding remarks

A.*Some Historic Remarks*: Theorems 3.1 and 3.2 are proved in [19]. In the region that the INS has smooth solutions, Theorem 3.1 was first proved in [8]. Theorem 4.1 is proved in [8], follows closely [13], where Theorem 4.1 is proved for the simpler dynamics, the asymmetric simple exclusion processes. Nonrevesible dynamics are closely related to the so-called nongradient dynamics. The first nontrivial solution of the fluctuation-dissipation equation (4) was given in [22] for Ginzburg-Landau models. Later on particle models are considered in [18]. These models [22, 18] are reversible, i.e., the generator L is symmetric. A nonreversible model is considered in [23], where the asymmetric part of the generator, $L - L^*$, is a bounded perturbation of the symmetric part. The resulting hydrodynamical equation is a nonlinear diffusion equation with no drift and thus cannot be used as models for the fluid. A first solution of the fluctuation-dissipation for truly nonreversible systems is given in [13], with some observation from [7].

B.*Further Results*:

1. The fluctuation and the scaling limit for the tagged particle of the lattice gas model in this review have not been fully investigated. The results on its toy mode, the asymmetric simple exclusion processes, are quite complete for dimension $d = 3$. It is proved [4] that the fluctuation in equilibrium for the asymmetric simple exclusion processes converges to an Ornstein-Uhlenbeck process with viscosity given by the bulk diffusivity in [7], and in [20] that the scaling limit of a tagged particle converges to a Brownian motion with a self-diffusion coefficient which is very different from the bulk diffusion coefficient. Notice all these results are for $d \geq 3$. The cases for $d = 1, 2$ remain open.

2. There are virtually no result on the compressible Navier-Stokes equations, For the asymmetric simple exclusion processes, the corresponding problem is partially answered in the smooth region in [12]. The zero-range dynamics is considered in [11].

3. Besides a solution to the fluctuation-dissipation equation, we need analytic information on the symmetric part of the generator to bridge the gap between the equilibrium theory and the

nonequilibrium one. Since our dynamics are conservative, the (symmetric part of the) generator has no spectral gap in infinite volume. But one still can ask the question as how the gap shrinks as the lattice becomes infinite, which indeed proceeds key information for our analysis. For the special case of symmetric simple exclusion processes, this is a well-known problem and many different proofs has been given (see, e.g., [18]). One actually needs a stronger inequality, the logarithmic Sobolev inequality inequality, proved in [24] in particular for the symmetric simple exclusion processes. Notice that due to the collision terms the symmetric part of the generator of the lattice gas model discussed in this review is different from the symmetric simple exclusion processes and there is no result either for the corresponding spectral gap or the logarithmic Sobolev inequality. Part of the reason is because [8] uses only the results for the symmetric simple exclusion processes.

3. So far we have restricted ourselves to the infinite temperature cases. The finite temperature dynamics, typically called the Kawasaki dynamics, have much richer structure as the invariant measures can have phase transitions. A curious fact for the Kawasaki dynamics is that there is no natural way to construct truly nonreversible dynamics with Gibbs states as invariant measures unless the dimension $d = 1$. Therefore, at present they are used as models for nonlinear diffusion equations and for studying the relations between the dynamics and the Gibbs States, but not as models for the fluid. Since the behavior of the spectral gaps changes completely when phase transitions occur, one typically needs to impose mixing assumptions to obtain spectral gap estimates [13], the logarithmic Sobolev inequality [25], or the hydrodynamical limit [26]. There is an interesting direction concerning the relaxation rates to the equilibrium. The goal is to prove an estimate of the form

$$\| P_t u - < u >_\mu \|^2_{L^2(\mu)} \leq C(u) t^{-d/2}$$

where $c(u)$ is some function or norm depending on u. The best result available [14] contains an extra $\log t$ factor in the estimate and the form of $c(u)$ is far from optimal.

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