I) Of all the n–gons with a given perimeter, which has the greatest area?

II) Determine $F(x)$, if for all real x and y, $F(x)F(y) - F(xy) = x + y$.

III) (a) Let R_n denote the number of ways of placing n non-attacking rooks on an $n \times n$ chessboard so that the resulting arrangement is symmetric about a 90° clockwise rotation of the board about its centre. Compute R_n.

(b) Let S_n denote the number of ways of placing n non-attacking rooks on an $n \times n$ chessboard so that the resulting arrangement is symmetric about the centre of the board. Compute S_n.

(c) Let T_n denote the number of ways of placing n non-attacking rooks on an $n \times n$ chessboard so that the resulting arrangement is symmetric about both diagonals. Compute T_n.

IV) Recall that an n-element permutation is a bijection

$$\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

An n-element derangement is an n-element permutation σ such that for every i, $\sigma(i) \neq i$.

(a) Let g_n be the number of n–elements derangements. Show that

$$g_1 = 0, \quad g_2 = 1 \quad \text{and} \quad g_n = (n-1)(g_{n-1} + g_{n-2}).$$

(b) Let f_n be the number of n–element permutations that have exactly one fixed point (namely, exactly one i such that $\sigma(i) = i$). Show that $|f_n - g_n| = 1$.

V) If the unit interval $[0, 1]$ is covered by a finite set of closed intervals S, prove that some subset of pairwise disjoint intervals in S has total length which is at least $\frac{1}{2}$.

VI) If a, b and c are lengths of 3 segments that can form a triangle, show that the same is true for $\frac{1}{a+b}$, $\frac{1}{b+c}$ and $\frac{1}{c+a}$.

VII) Determine all the straight lines lying in the surface $z = xy$.

VIII) Is it possible to choose ten different numbers from the set $\{0, 1, 2, \ldots, 14\}$ and place them in the ten circles in the figure below in a way that all positive differences between the pairs of numbers in adjacent circles (i.e., circles connected by an edge) are all to be different.