STEVO TODORČEVIĆ

COMPARING THE CONTINUUM WITH THE FIRST TWO UNCOUNTABLE CARDINALS

The purpose of this note is to survey some technical aspects of the two hypotheses $\mathfrak{c} = \aleph_1$ and $\mathfrak{c} = \aleph_2$ together with their joint negation $\mathfrak{c} > \aleph_2$. The reader is probably already puzzled by the fact that we decide to stop at the second uncountable cardinal rather than at some other value of the aleph–function. Well, there are two reasons for this. The first one is that presently there is a considerable lack of knowledge about statements like $\mathfrak{c} = \aleph_3$, $\mathfrak{c} = \aleph_4$, and so on. The other reason is that the present knowledge seems to suggest that indeed there is some "affinity" between $\mathfrak{c}$ and $\aleph_2$. It also seems that even a meager understanding of this affinity can be quite useful as we shall demonstrate this at the end of the article.

1. THE CONTINUUM HYPOTHESIS

This is perhaps one of the most widely known hypotheses of Set Theory. The root of its popularity is in its simplicity both as a statement and the way it is most frequently used. The existence of the very short well-ordering of the reals enables one to easily diagonalize and get some striking (or "highly implausible" as Gödel [11] calls them) objects such as (see [24]):

(C_1) An uncountable set of reals which has countable intersection with every nowhere dense set of reals.

(P_3) An uncountable subset X of the Hilbert cube $[0,1]^{\mathrm{N}}$ such that every uncountable $Y \subseteq X$ is projected onto the interval $[0,1]$ by all but finitely many of the projections.

(P_{10}) An infinite-dimensional subset of the Hilbert cube which contains no uncountable finite-dimensional subset.

The statement P_{10}, originally due to Hurewicz [15], is not only a consequence of CH but it also implies CH since any metric space of size $< \mathfrak{c}$ is 0-dimensional. The statement P_3 is another equivalent of CH. Proving this statement from CH did cause a considerable difficulties (see [24; pp. 189–193]) and we conjecture that this is so because of a too strong preoccupation with getting a statement equivalent to CH rather than just its consequence (like C_1). In fact, P_3 is so made (perhaps artificially) that the implication $P_3 \rightarrow$CH is a triviality. Relaxing P_3 by replacing the interval $[0,1]$ by the set of all countable ordinals makes it a rather direct consequence of C_1 (see [25; §6]) which itself is a rather simple consequence of CH. One thing one notices while studying the known consequences of CH is that they all can be reformulated to be

Σ^2_1-statements, i.e. statements asserting the existence of a set of reals whose properties are describable using only quantification over the reals themselves. For example P_3 says that there is a mapping $p: \omega_1 \times \omega_1 \to \mathrm{R}$ such that for every countable $A \subseteq \omega_1$ and $f: A \to \mathrm{R}$ there is γ in ω_1 such that for all β in ω_1 above γ there exists α in A such that $p(\alpha, \beta) = f(\alpha)$. The Σ^2_1-form of the statement P_{10} says that there is a subset of the Hilbert cube which has countable intersection with every 0-dimensional G_δ-subset of the cube. The following remarkable result of Woodin [32] (proved using a certain large-cardinal hypothesis) not only gives the explanation of this phenomenon but it also points out how one should look at any future application of CH.

Theorem 1 *If a Σ^2_1-statement is true in one forcing extension of the universe then it is also true in any other forcing extension which satisfies CH.*

Thus, if there is a statement for which we would like to know whether it is implied by CH or not then the first thing to do is to try to restate it as a Σ^2_1-statement. If this is possible then the statement is almost certainly a consequence of CH for we shall never be able to construct a forcing extension which would show that this is not so.

2. CODING THE REALS

The increasing sophistication in the area of iterated forcing resulted in a profusion of models of Set Theory built for the purpose of proving the independence of statements like Souslin's Hypothesis, Borel's Conjecture, and so on. It turns out that many such proofs, especially the deeper ones involving other than the finite support iterations, could not produce models where the continuum would have value different from $\aleph_2$. For example, a typical such proof is Laver's consistency proof ([19]) of Borel's Conjecture. It is therefore natural to ask whether the statements in question do pose some restriction on the continuum such as $\mathfrak{c} = \aleph_2$ or not. This kind of question is especially interesting when the considered statements are some kind of maximality principles, such as, for example, various forcing axioms asserting that we can have a sufficiently generic filter for every member of a large class of posets. (See the footnote 23 of [11] regarding this point.) The purpose of this and the next section is to explain a pair of such maximality principles which jointly imply that the continuum is equal to $\aleph_2$ and which are jointly true in a large class of models discussed above and especially in those made for the purpose of proving the consistency of certain forcing axioms which, in their complexity, are just above the Martin's Axiom (which itself does not pose any restriction on the size of the continuum except its regularity).

Let N^{N} denote the structure of all sequences of integers ordered by the ordering $<^*$ of eventual dominance. Define osc: $(\mathrm{N}^{\mathrm{N}})^2 \to$Card by

$$\mathrm{osc}(a, b) = \#\{n \in \mathrm{N} : a(n) \leq b(n) \text{ and } a(n+1) > b(n+1)\}.$$

This is a quite interesting object whose properties are closely tied to the notion of unboundedness in N^{N} under $<^*$ (see [26], [27]). For example, it has the property that a subset of N^{N} which is at the same time σ-directed and unbounded under $<^*$ realizes all possible finite oscillations. Its properties can be summarized as follows where

$$o : (\mathrm{N}^{\mathrm{N}})^2 \to \mathrm{N}$$

is obtained by composing osc with some natural (recursive) mapping from N into N (see [26], [27; §1]):

Theorem 2 *Suppose A is a σ-directed unbounded subset of N^{N} and that X is an unbounded subset of $A^{[k]}$ for some integer $k \geq 1$. Let D be a countable dense subset of X. Then there is $b \in A$ such that for every integer $n \geq 1$, every $h\colon k \times n \to \mathrm{N}$, and every $x \in A^{[n]}$ above b there is $d \in D$ such that $o(d_i, x_j) = h(i, j)$ for all $i < k$ and $j < n$.*

(Here $A^{[k]} = \{x \in A^k : x_i \neq x_j \text{ for all } i \neq j < k\}$.) In simple words, this result tells that osc and o have a remarkably rich behavior on any A which is σ-directed and unbounded in N^{N}. A typical such A would be an unbounded $W \subseteq \mathrm{N}^{\mathrm{N}}$ which is well-ordered by $<^*$. So let us fix such a W. For x in W let x^{+n} be its nth successor in W. A subset $X \subseteq W$ is *σ-complete* if for every increasing sequence $\{x_n\}$ of elements of X the least upper bound of $\{x_n\}$ in W is also a member of X.

Definition 1 *A set $X \subseteq W$* codes *a "real" $r \in \mathrm{N}^{\mathrm{N}}$ if for every n there is a countable decomposition $X = \bigcup_{i=1}^{\infty} X_i^n$ such that*

$$o(x^{+n}, y^{+n}) = r(n)$$

for all $x <^* y$ in X_i^n and $i \in \mathrm{N}$.

Let PCR_W be the statement that every real is coded by a σ-complete subset of W with no last element.

Lemma 1 *PCR_W implies that there is a 1–1 mapping from* R *into W.*

Proof: For $r \in \mathrm{R}$ $(= \mathrm{N}^{\mathrm{N}})$ fix a σ-complete $X_r \subseteq W$ which codes r. Note that if $r \neq s$ then X_r and X_s have different supremums in $W \cup \{\infty\}$

The Principle of Coding the Reals, PCR, which would state that PCR_W is true for every unbounded and well-ordered $W \subseteq \mathrm{N}^{\mathrm{N}}$ is an interesting hypothesis. To see what it tells us about the continuum let $\mathfrak{b}$ be the minimal order-type of any such $W \subseteq \mathrm{N}^{\mathrm{N}}$. This is a well-known cardinal characteristic of the continuum whose importance has been recognized very early (see [6]). It is clearly a regular uncountable cardinal $\leq \mathfrak{c}$. From Lemma 1 we have the following

Proposition 1 *PCR implies that $\mathfrak{c} = \mathfrak{b} > \aleph_1$.*

Proof: Note that the equality $\mathfrak{b} = \aleph_1$ is impossible because in a well-ordered set of type ω_1 all σ-complete subsets with no least elements must be unbounded, so only one real can be coded.

The statement PCR is true in any situation where we can do some internal forcing. In fact, Theorem 2 is exactly saying that the poset of finite approximations to a set which codes a real is very well-behaved which can be stated in the following precise way.

Lemma 2 *Suppose $C \subseteq W$ is σ-complete and unbounded and that its order-type is a regular (uncountable) cardinal θ. Then for every real r the poset of finite approximations to the sequence of decompositions $C = \bigcup_{i=1}^{\infty} C_i^n$ $(n \in \mathrm{N})$ as in Definition 1 is a θcc poset.*

Proof: Clearly we may assume C consists only of limit points of W. For $n \in \mathrm{N}$ let C^{+n} be the set of all c^{+n} for c in C. Let C^* be the union of C^{+n}'s. Conditions of the poset $\mathcal{P}$ are finite structures which tell us some finite information about the decomposition of C. In particular, for each $p \in \mathcal{P}$ the set S_p of all elements of C^* which p mentions is finite. We have to show that every $\mathcal{F} \subseteq \mathcal{P}$ of size θ contains two compatible conditions p and q. It is not hard to see that we may assume that the elements of $\mathcal{F}$ have disjoint supports, and moreover that they all are isomorphic finite structures. This in particular means that their supports are of the same size k and that we may identify them with the corresponding elements of $(C^*)^{[k]}$ which enumerate them increasingly according to the well-ordering on W. Let X be the subset of $W^{[k]}$ thus obtained and let D be a countable dense subset of it. Let $b \in C$ be the bound given by Theorem 2. The disjointness of supports and the assumption on the order-type of C imply that we can fix a $q \in \mathcal{F}$ such that $x = S_q$ is above b. Now, to find $d = S_p$ in D such that p and q are compatible, the oscillation map o has to satisfy the requirements of Definition 1 on the union of S_p and S_q, i.e. certain requirements of the form $o(u, v) = r(n)$ for $u \in S_p$ and $v \in S_q$. Given the fact that members of $\mathcal{F}$ are isomorphic structures it is easy to find $h: k \times k \to \mathrm{N}$ such that if $d = S_p$ from D and $x = S_q$ satisfy the conclusion of Theorem 2 for this h then the requirements are met.

Thus, if we start with a fixed W and a real r and if we go to a forcing extension of the universe obtained by simply collapsing the continuum to $\aleph_1$ by the natural σ-closed poset then all the properties of the oscillation maps osc and o (as well as of the set W) remain undisturbed. This indestructibility is one property of osc not shared by any other known two-place function and it is very crucial to us at this point. In the forcing extension, we can easily find $C \subseteq W$ as in Lemma 2 of order-type ω_1 so the corresponding poset of finite approximations is ccc. Thus if we are able to internally force with a σ-closed poset (in fact only the standard one which collapses $\mathfrak{c}$ to $\aleph_1$) followed by a ccc poset, then we can prove that PCR is true. It turns out that the forcing axiom for σ-closed$*$ccc posets is one of the weakest known and most frequently used extensions of MA. The reader is referred to [1], [2], [3], [5], [9], [23], [27] and [31] for more about this kind of axioms.

3. OPEN COLORING AXIOM

For a set of reals (or more generally for a topological space X) let OCA_X denote the following regularity property: For every open symmetric relation $K \subseteq X^{[2]}$ either

1. there is a countable decomposition $X = \bigcup_{n=1}^{\infty} X_n$ such that $X_n^{[2]} \cap K = \emptyset$ for all n, or
2. there is an uncountable $Y \subseteq X$ such that $Y^{[2]} \subseteq K$.

Note that OCA is preserved under continuous images and that if in (ii) we require Y to be a perfect set (homeomorphic to the Cantor set) we get a statement, call it OCA_X^*, which can be considered as a natural 2-dimensional analogue of the classical Perfect-Set Property (see [16; 3.1]). Note also that this stronger form OCA_X^* is true for some simple X such as the reals or the irrationals and therefore for all analytic sets of reals. It turns out that the requirement that K is open rather than, say, closed is crucial even when $X = \mathrm{R}$ (see [28]). Let OCA be the statement that OCA_X is true for every separable metric space X, or equivalently, for every set of of reals X. This has proved to be a very useful axiom for a wide range of problems (see [27]). Its most surprising consequence is the following fact discovered in [27] during the course of proving certain results about gaps in the structure $(\mathrm{N}^{\mathrm{N}}, <^*)$.

Proposition 2 *OCA implies that* $\mathfrak{b} = \aleph_2$.

A gap in $(\mathrm{N}^{\mathrm{N}}, <^*)$ is a pair (A, B) of subsets of N^{N} such that $A \cup B$ is a chain under $<^*$ and $a <^* b$ for all $a \in A$ and $b \in B$, but there is no $x \in \mathrm{N}^{\mathrm{N}}$ such that $a <^* x <^* b$ for all $a \in A$ and $b \in B$. If α is the cofinality of $(A, <^*)$ (i.e., the minimal order-type of a subset of A cofinal in A) and if β is the cofinality of $(B, >^*)$ then we call (α, β^*) the *type of* (A, B). Gaps of type (ω_1, ω_1^*) are called *Hausdorff gaps* because Hausdorff was the first to show their existence ([13], [14]). Gaps of type (θ, ω^*) or (ω, θ^*) are called *Rothberger gaps.* Their existence was also first proved by Hausdorff [13], and later rediscovered by Rothberger [22]. For example, they showed that there is always an $(\mathfrak{b}, \omega^*)$-gap in $(\mathrm{N}^{\mathrm{N}}, <^*)$. They also showed that if there is a (θ, ω^*) gap then there is an unbounded θ-chain in $(\mathrm{N}^{\mathrm{N}}, <^*)$. The following Lemma about the influence of OCA on gaps first appeared in [27; §8].

Lemma 3 *If OCA is true then every gap* (A, B) *in* $(\mathrm{N}^{\mathrm{N}}, <^*)$ *is either Hausdorff or Rothberger.*

Proof: Let A^* be set of all elements obtained by taking all finite changes of elements of A and let B^* be obtained from B in the same way. Let $A^* \otimes B^*$ be the set of all $(a, b) \in A^* \times B^*$ such that $a(n) \leq b(n)$ for all n. Let $X = A^* \otimes B^*$ considered as a subspace of the square of the Baire space N^{N}. Let $K \subseteq X^{[2]}$

be the set of all pairs $\langle\langle a,b\rangle,\langle a',b'\rangle\rangle$ from $X^{[2]}$ such that there is an $n\in\mathrm{N}$ for which $a(n)>b'(n)$ or $a'(n)>b(n)$. The alternatives of OCA give us the conclusion of Lemma 3.

The following Lemma also appears in [26] (as Lemma 3.10).

Lemma 4 *If $\mathfrak{b}>\aleph_2$ then there is a gap (A,B) in $(\mathrm{N}^{\mathrm{N}},<^*)$ which is neither Hausdorff nor Rothberger.*

Proof: Pick a subset A of N^{N} which has order-type ω_2 under $<^*$ and extend it to a maximal $<^*$-totally ordered subset L of N^{N}. By the assumption of $\mathfrak{b}>\aleph_2$, the set A is not cofinal in L so let B be the upper half of the gap in L which A determines. Then (A,B) is also a gap in $(\mathrm{N}^{\mathrm{N}},<^*)$. Note that B does not have a minimal element for if $b=\min(B)$ then $b-1$ would be a member of N^{N} which could be added to L contradicting its maximality. This phenomenon is specific for the structure $(\mathrm{N}^{\mathrm{N}},<^*)$ and it is crucial for us here. Our assumption $\mathfrak{b}>\aleph_2$ and the already quoted result of Hausdorff [13] and Rothberger [22] imply that the coinitiality of B must be uncountable. So (A,B) is neither a Hausdorff nor a Rothberger gap.

It should be noted that we had a considerable difficulty in getting the right form of Lemma 4 which may not be so visible from its relatively short proof. The difficulty was in getting rid of the influence of a very similar result, proved long time ago by a number of people including the author ([2; Theorem 4.4]), about the structure $(\mathcal{P}(\mathrm{N}),\subseteq^*)$ where the place of $\mathfrak{b}$ is played by the cardinal $\mathfrak{t}$, the minimal order-type of a nonextendable tower of subsets of N. [A *nonextendable tower* is a set W of coinfinite subsets of N which is well-ordered by $\subseteq^*$ and which cannot be end-extended.] The older result, however, is quite useless in the present context because there is no oscillation theory of the structure $(\mathcal{P}(\mathrm{N}),\subseteq^*)$ where the corresponding form of Theorem 2 is true for A a nonextendible tower. (See [30] for an explanation of this point.) To prove Proposition 2 we still owe the reader the explanation of why OCA implies $\mathfrak{b}>\omega_1$. Well, assume $\mathfrak{b}=\omega_1$ and choose well-ordered unbounded $A\subseteq\mathrm{N}^{\mathrm{N}}$ of type ω_1. Let $K\subseteq A^{[2]}$ be the set of all pairs (a,b) such that $\operatorname{osc}(a,b)>0$. Then K is open and the fact that every unbounded $X\subseteq A$ realises oscillation 0 means that OCA is false for A and K. We finish this section with the remark that similarly to PCR considered in the last section, OCA is a consequence of the forcing axiom for the class of posets which can be written in the form σ-closed$*$ccc (se [27; §8]).

4. COVERING THE REALS WITH STRONG MEASURE ZERO SETS

The well-known unpublished manuscript of Gödel [12], written before the great sophistication in iterated forcing discussed above, contains interesting ideas which might be useful in explaining $\mathfrak{c}=\aleph_2$ in some of the models not covered by §§2 and 3. So the purpose of this section is to draw the reader's attention to some of these ideas. Gödel's approach was to cover the reals with a small family of "small sets" of reals. The "small sets" are sets of strong measure zero in the sense of E. Borel [4], i.e. the subsets of R which can be covered by a sequence $\{I_n\}$ of intervals of any prescribed sequence of lengths $\{\epsilon_n\}$. As Gödel has realized, these sets are in close relation with the structures R^{N} or N^{N} when ordered by any variation of the ordering of eventual dominance. In particular, the dominating number $\mathfrak{d}$, defined as the minimal cardinality of a subset D of N^{N} which is cofinal in $(\mathrm{N}^{\mathrm{N}}, <^*)$, should play the important role. For example, note that every strong measure zero set is included in a $G_{\mathfrak{d}}$-set of strong measure zero, i.e. in a strong measure zero set which is equal to the intersection of a family of no more than $\mathfrak{d}$ open sets of reals. In fact, even larger powers seem to be relevant as the following fact shows. [Here SMZ stands for the ideal of strong measure zero sets of reals ordered by the inclusion while $\mathcal{Z}$ stands for the family of all subsets of N of asymptotic density 0 (i.e. sets $A \subseteq \mathrm{N}$ with the property that $\lim_{n\to\infty} \#(A \cap \{1, \ldots, n\})/n = 0$) ordered by the relation $\subseteq^*$ of inclusion modulo the ideal of finite sets, and where $\mathcal{Z}^{\mathfrak{d}}$ is the set of all functions from the initial ordinal $\mathfrak{d}$ into $\mathcal{Z}$ ordered coordinatewise.]

Proposition 3 *There is a monotonic map $\Phi\colon \mathcal{Z}^{\mathfrak{d}} \to \mathrm{SMZ}$ whose range is cofinal in SMZ.*

Proof: Let $\{q_n\}$ be an enumeration of the rationals and fix a dominating $D \subseteq \mathrm{N}^{\mathrm{N}}$ of size $\mathfrak{d}$. Choose $F\colon \mathrm{SMZ} \to \mathcal{Z}^D$ so that

$$X \subseteq \textstyle\bigcap_{m=1}^{\infty} \bigcup \{(q_n - 1/d(n), q_n + 1/d(n)) : n \in F(X)(d),\ n \geq m\}$$

for all $X \in \mathrm{SMZ}$ and $d \in D$. For $z \in \mathcal{Z}^D$, set $\Phi(z) = \bigcup \mathcal{F}_z$, where

$$\mathcal{F}_z = \{X \in \mathrm{SMZ} : F(X)(d) \subseteq^* z(d) \text{ for all } d \in D\}.$$

It is easily checked that Φ is as required.

The reader interested in more information about objects such as SMZ, $\mathcal{Z}$, N^{N} is referred to [10] which gives an excellent synthesis of results of this sort. One of the crucial points of Gödel's approach rests on a proper understanding of Perfect-Set Properties of the following form:

Problem 1 *When one can conclude that every $G_{\mathfrak{d}}$-set of reals of size $> \mathfrak{d}$ (or $\mathfrak{d}^+$ if this makes a difference) contains a nonempty perfect subset?*

This is so because in any situation where $G_{\mathfrak{d}}$-sets of reals have this kind of a Perfect-Set Property, the size of the continuum reduces to the covering number of the ideal SMZ, i.e. to the minimal cardinality of a family $\mathcal{F} \subseteq \mathrm{SMZ}$ such that $\bigcup \mathcal{F} = \mathrm{R}$. (Note the relevance of Borel's Conjecture in this context.) One way of understanding this number might be through the study of powers like $\mathcal{Z}^\partial$, but more likely the real information will come from a deeper study of SMZ itself.

The idea of using strong measure zero sets to cover the reals in order to get some information about the size of the Continuum can perhaps be transfered to other notions of smallness as far as no perfect set of reals satisfies it. A natural candidate would be the closely related notion of an universal measure zero set of reals. These are the sets of reals which get value zero under any nonatomic Borel measure on R. They are not as small as the sets of strong measure zero since the analogue of Borel's Conjecture is false for this family, i.e. there is always an uncountable universal measure zero set of reals. This means that it might be easier to avoid the phenomenon encountered above where Borel's Conjecture trivializes any attempt to cover R with a small family of strong measure zero sets.

5. WHEN THE CONTINUUM IS BIGGER THAN $\aleph_2$

For an integer n a family $\mathcal{F}$ of sets is *n-disjoint* if it can be decomposed in no more than n disjoint subfamilies. It is clear that by compactness $\mathcal{F}$ is n-disjoint iff every finite subfamily of $\mathcal{F}$ is n-disjoint. In 1946 Richard Rado [21] was able to considerably improve this for certain families of sets $\mathcal{F}$. He proved that a family of intervals (or more generally convex sets) $\mathcal{F}$ of some fixed linearly ordered set is n-disjoint iff every subfamily of $\mathcal{F}$ of size at most $n+1$ is n-disjoint. This kind of compactness is a prominent theme in the field of Combinatorics. For example, the famous Dilworth chain decomposition theorem is also a result of this sort: a partially ordered set $\mathcal{P}$ can be covered by a family of $\leq n$ chains iff this is true for every $\mathcal{P}_0 \subseteq \mathcal{P}$ of size at most $n+1$. The pleasing behavior of families of convex subsets of linearly ordered sets has led Rado to conjecture the similar behavior at the next level:

(RC) A family of convex subsets of a linearly ordered set is σ-disjoint iff each of its subfamilies of size at most $\aleph_1$ is σ-disjoint.

(Here "σ-disjoint" means "the union of countably many disjoint subfamilies".) This turns out to be a very interesting compactness principle unique in the sense that it works at the level where any other known compactness principle fails badly. Its relevance to the question about the size of the continuum is revealed in the following fact proved in [29].

Theorem 3 *To every sequence $e_\alpha: \alpha \to |\alpha|$ $(\alpha < \omega_2)$ of 1–1 mappings we can associate a family $\mathcal{F}(\vec{e})$ of convex subsets of some linearly ordered set such that*

1. *$\mathcal{F}(\vec{e})$ can be written as the union of an increasing sequence $\mathcal{F}_\xi(\vec{e})$ $(\xi < \omega_2)$ of σ-disjoint subfamilies,*
2. *if $\mathcal{F}(\vec{e})$ itself is σ-disjoint then* $\mathrm{R} \subseteq L[\vec{e}]$.

(Here $L[\vec{e}]$ is the constructible subuniverse generated by the sequence $\vec{e}$.) This means that RC implies $\mathfrak{c} \leq \aleph_2$. (It is interesting that RC does not decide whether $\mathfrak{c} = \aleph_1$ or $\mathfrak{c} = \aleph_2$.) It follows that while the negation of CH, $\mathfrak{c} > \aleph_1$, is extremely poor in its consequences as far as "striking objects" are concerned this cannot be said about its apparently slight strengthening $\mathfrak{c} > \aleph_2$. The assumption $\mathfrak{c} > \aleph_2$ gives us a very striking object indeed, a family $\mathcal{F}$ of size $\mathfrak{c}$ of convex subsets of some linearly ordered set which is not σ-disjoint but every subfamily $\mathcal{F}_0$ of $\mathcal{F}$ of size at most $\aleph_1$ is σ-disjoint. We refer the reader to our paper [29] for further information about RC which we choose not to include here. Instead we chose to give an application of these ideas to an old problem of Erdös and Hajnal [7]:

Proposition 4 *For every decomposition of the family* N^{ω_2} *of all integer-valued functions defined on* ω_2 *into countably many pieces one of the pieces must contain two almost disjoint functions.*

(Two functions $f, g \in \mathrm{N}^{\omega_2}$ are *almost disjoint*, written as $f \perp g$, if there is $\beta < \omega_2$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \beta$.) In other words, the graph $\mathcal{G}_{\omega_2} = (\mathrm{N}^{\omega_2}, \perp)$ is not countably chromatic which is the way Erdös and Hajnal have stated their problem. A close examination of the graph $\mathcal{G}_{\omega_2}$ reveals that this is an interesting object sensitive to many properties of the second uncountable cardinal. For example, note that $\mathrm{Chr}(\mathcal{G}_{\omega_2})$ is bounded by the size of any ultrapower of the form $\mathrm{N}^{\omega_2}/\mathcal{U}$ where $\mathcal{U}$ is a uniform ultrafilter on ω_2. The sizes of ultrapowers have been a reoccuring theme in Set Theory as well as in Model Theory. We mention the recent remarkable result of Foreman [8] which has produced a model of Set Theory (assuming the consistency of a certain large cardinal axiom) in which the minimal size of ultrapowers of the form $\mathrm{N}^{\omega_2}/\mathcal{U}$ for $\mathcal{U}$ a uniform ultrafilter on ω_2 is equal to $\aleph_1$. So, in particular in this model $\mathrm{Chr}(\mathcal{G}_{\omega_2})=\aleph_1$ which we now know to be the minimal possible size. Another interesting property of $\mathcal{G}_{\omega_2}$ is that each of its subgraphs of size $\aleph_1$ is countably chromatic and that $\mathcal{G}_{\omega_2}$ contains a copy of every graph of size $\aleph_2$ which has this property. Thus, $\mathcal{G}_{\omega_2}$ is sensitive to any kind of incompactness of the chromatic number at this level. This fact was used by Baumgartner, Shelah, Komjath and others to show that in models of Set Theory in close proximity to the constructible universe the chromatic number of $\mathcal{G}_{\omega_2}$ can be rather large and in particular can have the maximal possible value $2^{\aleph_2}$. The sensitivity of $\mathcal{G}_{\omega_2}$ to almost any kind of incompactness has led us to discover the proof of Proposition 4 by connecting it with Theorem 3. The proof of Proposition 4 spits into two cases:

Case I: The power of the continuum is at most $\aleph_2$. This case is essentially covered already in the Erdös–Hajnal paper [7] who state their result assuming CH. That their argument also covers the case $\mathfrak{c} = \aleph_2$ was, according to Hajnal, first noticed by P. Komjath. Here is a Lemma whose hypothesis is true in both cases $\mathfrak{c} = \aleph_1$ and $\mathfrak{c} = \aleph_2$ and whose proof is essentially a repetition of the old Erdös–Hajnal argument.

Lemma 5 *If there is a sequence* $g_\alpha : \omega_2 \to \mathrm{R}$ $(\alpha < \mathfrak{c}^+)$ *of pairwise almost disjoint functions then* $Chr(\mathcal{G}_{\omega_2})$ *is uncountable.*

Proof: For $\alpha < \beta < \mathfrak{c}^+$ define $f_{\alpha\beta} \in \mathrm{Z}^{\omega_2}$ by

$$f_{\alpha\beta}(\xi) = \Delta(f_\alpha(\xi), f_\beta(\xi)) \cdot \mathrm{sgn}(f_\alpha(\xi) - f_\beta(\xi))$$

where for two reals x and y, $\Delta(x, y)$ is the minimal n such that the nth rational number is in between x and y (for some fixed enumeration $\{q_i\}$ of Q). Then $f_{\alpha\beta} \perp f_{\beta\gamma}$ for all $\alpha < \beta < \gamma < \mathfrak{c}^+$, so we are done by $\mathfrak{c}^+ \to (3)^2_\omega$.

Case II: The power of the continuum is bigger than $\aleph_2$. Consider the family $\mathcal{F}(\vec{e})$ from Theorem 3 and for each $\xi < \omega_2$ fix a σ-disjoint decomposition $g_\xi: \mathcal{F}_\xi(\vec{e}) \to \mathrm{N}$. This enables us to define

$$\Phi: \mathcal{F}(\vec{e}) \to \mathrm{N}^{\omega_2}$$

by the following rules:

(a) $\Phi(I)(\xi) = 1$ for $I \notin \mathcal{F}_\xi(\vec{e})$,

(b) $\Phi(I)(\xi) = g_\xi(I)$ for $I \in \mathcal{F}_\xi(\vec{e})$.

Then $I \cap J \neq \emptyset$ implies $\Phi(I) \perp \Phi(J)$ for every $I, J \in \mathcal{F}(\vec{e})$, so the conclusion of Proposition 4 follows from the fact that under $\mathfrak{c} > \aleph_2$ the family $\mathcal{F}(\vec{e})$ is not σ-disjoint.

We refer the reader to a projected paper of A. Hajnal and the author which will contain information about the chromatic number of $\mathcal{G}_\theta = (\mathrm{N}^\theta, \perp)$ for some other values of θ.

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Matematički Institut
Kneza Mihaila 35
11000 Beograd, PP 367
Yugoslavia

Department of Mathematics
University of Toronto
Toronto, Canada
M5S 1A1