MAT 246S **Solutions to Term Test** **March 9, 2011**

(1) (10 pts) Find the formula for the sum $1\cdot 2-2\cdot 3+3\cdot 4-\ldots+(2n)\cdot(2n-1)-(2n)\cdot(2n+1)$ and prove it by mathematical induction.

Solution

Observe that $(2n)(2n-1)-(2n)(2n+1)=(2n)\cdot(-2)=-4n$ Thus we need to find $-4\cdot 1-\ldots-4n=-4(1+\ldots n)=-4\frac{n(n+1)}{2}=-2n(n+1)$.

We prove this by induction.

When $n=1$ we have $1\cdot 2-2\cdot 3=2-6=-4=-2\cdot(1)\cdot(2)=-4$.

Induction step. Suppose $1\cdot 2-2\cdot 3+3\cdot 4-\ldots-(2n)\cdot(2n+1)=-2n(n+1)$ then $1\cdot 2-2\cdot 3+3\cdot 4-\ldots-(2n)\cdot(2n+1)+(2n+1)\cdot(2n+2)-(2n+2)\cdot(2n+3)=-2n(n+1)+(2n+1)\cdot(2n+2)-(2n+2)\cdot(2n+3)=-2n(n+1)-2(2n+2)=-2(n+1)(n+2)$.

(2) (10 pts) Find the remainder when 6^{100} is divided by 14.

Solution

First we observe that $6\equiv -1 \pmod 7$. Hence $6^{100}\equiv(-1)^{100}=1 \pmod 7$. Thus $6^{100}\equiv 1 \pmod 7 \equiv 8 \pmod 7$. This means that 7 divides $6^{100}-8$. But $6^{100}-8$ is even and 2 also divides $6^{100}-8$. Since $(2,7)=1$ this means that 14 divides $6^{100}-8$, i.e. $6^{100}\equiv 8 \pmod{14}$.

Answer: 8.

(3) (10 pts) Find the integer a, $0\le a<37$ such that $(34!)a\equiv 1 \pmod{37}$.

Solution

Since 37 is prime, by Wilson's theorem, $36!\equiv -1 \pmod{37}$.

We rewrite $34!\cdot 35\cdot 36 \equiv= -1 \pmod{37}$. Since $36\equiv -1 \pmod{37}$ this gives $34!\cdot 35\equiv 1 \pmod{37}$.

Answer: $a=35$.

(4) (10 pts) Find two different integer solutions of the equation

$$34x + 50y = 2$$

Solution

First we simplify $17x + 25y = 1$. Note that $(17, 25) = 1$. We compute it using Euclidean algorithm.

$25 = 1 \cdot 17 + 8, 17 = 2 \cdot 8 + 1$. Hence $8 = 25 \cdot 1 - 17 \cdot 1$ and $1 = 17 \cdot 1 - 2 \cdot 8$. plugging in the former equation into the latter we get $1 = 17 \cdot 1 - 2(25 \cdot 1 - 17 \cdot 1) = 17 \cdot 3 - 25 \cdot 2$. Hence $x_0 = 3, y_0 = -2$ is one solution. To get another solution recall that if x_0, y_0 solves $ax + by = c$ then $x = x_0 + kb, y = y_0 - ka$ solves $ax + by = c$ for any integer k. Setting $k = 1$ we get $x_1 = 3 + 25 = 28, y_1 = -2 - 17 = -19$ is another solution.

Answer: $x_0 = 2, y_0 = -3$ and $x_1 = 28, y_1 = -19$.

(5) (10 pts) Find all rational roots of the equation $x^3 + x + 2 = 0$.

Solution

Let $c = \frac{p}{q}$ be arational solution of $x^3 + x = 2$ where $(p, q) = 1$. By a theorem from class this implies that $p|2$ and $q|1$. So the only options for x are ± 1 and ± 2. Plugging in those numbers into the equation we get. If $x = 1$ then $x^3+x+2 = 4 \neq 0$. If $x = -1$ then $x^3+x+2 = -1-1+2 = 0$. If $x = 2$ then $x^3+x+2 = 8+2+2 = 12 \neq 0$ and if $x = -2$ then $x^3 + x + 2 = -8 - 2 + 2 = -8 \neq 0$.

Hence, the only rational solution is $x = -1$.

Answer: $x = -1$ is the only rational root.