(1) Finish the proof of the Shroeder-Berenstein Theorem from class.

Recall that the proof goes as follows. Let $f\colon S \to T$ and $g\colon T \to S$ be 1-1. Define S_S to be the set of points in S whose last ancestor is in S. Define S_T to be the set of points whose last ancestor is in T. Define S_∞ to be the set of points in S which have infinitely many ancestors. Define T_S, T_T, T_∞ similarly.

Then define $h\colon S \to T$ by the formula

$$h(s) = \begin{cases} f(x) \text{ if } s \in S_S \cup S_\infty \\ g^{-1}(s) \text{ if } s \in S_T \end{cases}$$

It was shown in class that $h\colon S_S \to T_S$ is 1-1 and onto.

Finish the proof by showing that $h\colon S_T \to T_T$ and $h\colon S_\infty \to T_\infty$ are also 1-1 and onto.

(2) Let S be infinite and $A \subset S$ be finite.

Prove that $|S| = |S \backslash A|$.

(3) Let $S = (0, 1)$ and $T = [0, 1)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = \frac{x+1}{2}$.

(a) Find $S_S, S_T, S_\infty, T_S, T_T, T_\infty$

(b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Schroeder-Berenstein theorem.

(4) Let $S = P(\mathbb{N})$

(a) Show that $|S| \leq |\mathbb{R}|$.

Hint: represent a subset A of N as a sequence of 1s and 0s such that the n-th element of the sequence is 1 if $n \in A$ and is 0 if $n \notin A$.

(b) Show that $|S| = |\mathbb{R}|$

Hint: It was shown in class that $|\mathbb{R}| \leq |P(\mathbb{N})|$.

(5) (a) Let T be an infinite set and let S be countable.

Prove that $|T \cup S| = |T|$

(b) Let T be the set of all transcendental numbers.

Prove that $|T| = |\mathbb{R}|$.

Hint: use part a).