A UNIVERSAL PARABOLIC MAP

A. EPSTEIN AND M. YAMPOLSKY

Of all the beautiful developments in modern complex dynamics, perhaps the most fascinating is one-dimensional renormalization theory. This theory serves to explain the universality phenomena, such as the celebrated Feigenbaum-Coullett-Tresser universality in transition to chaos in unimodal maps, and is also a rich source of results about the geometry of parameter spaces of analytic maps. There are two branches of the renormalization theory in one-dimensional dynamics, one dealing with unimodal maps, the other with critical homeomorphisms of the circle. Renormalization became a part of complex dynamics mainly through the work of Sullivan ([Sul1, Sul2, MvS]) and Douady-Hubbard [DH2], following an important first step taken earlier by H. Epstein [E, EE]. We will not attempt to give a survey of these results here; the interested reader is referred to the papers of Lyubich [Lyu3, Lyu4] for an account of the unimodal renormalization theory (of which these papers are the culmination), and to the paper of one of the authors [Ya3] for a description of the theory of critical circle maps. The two theories are closely intertwined, but possess some interesting differences, some of which are discussed in the latter reference.

This paper is concerned with a very special case of renormalization in one-dimensional dynamics, the so-called *parabolic renormalization*. This construction arises as a degenerate case of the usual renormalization, when the periods of the renormalized maps become infinite. The first mention in the literature of the universality phenomenon associated to it was in the work of Lanford [Lan1, Lan2], who incorporated the parabolic renormalization of critical circle maps into the general renormalization framework. Similarly to the usual renormalization, the parabolic renormalization yields various results of geometric nature. In [BDDS] it is used to explain the occurence of certain sequences of small copies of the Mandelbrot set $\mathcal{M}$ inside $\mathcal{M}$. More significantly, it features prominently in the celebrated article of Shishikura [Sh] on the Hausdorff dimension of the boundary of $\mathcal{M}$. The "thick" quadratic Julia sets with Hausdorff dimension almost 2 are created in that paper by applying the second iteration of a parabolic renormalization procedure to a quadratic polynomial with a parabolic periodic orbit.

In the setting of critical circle maps most of the conjectures on the parabolic renormalization were resolved by one of the authors in [Ya2]. The parallel questions in the unimodal setting were treated by Hinkle in [Hin]. However, in [Ya3] a new approach to the renormalization of critical circle mappings was presented to handle the hyperbolicity problem. The classical (i.e. twenty years old) approach is to define the renormalization in the language of *critical commuting pairs*. This is a forced measure due to a principal difficulty in specifying

Date: November 17, 2002.

a well-defined renormalized circle map. This difficulty was finessed in [Ya3]. The resulting definition easily extends to the parabolic case (and was previously known in this setting); due to its naturality, this definition radically simplifies the existing theory. In this paper we reprove the relevant results of [Ya2] and extend them to completely settle Lanford's conjectures for the parabolic renormalization of critical circle maps. This new renormalization theory is notably simplier than the usual one. One of the most technical aspects, the complex *a priori* bounds, is replaced by an elementary argument of C. Petersen [Pet]. More importantly, the *renormalization towers* introduced by McMullen in [McM2] in this context become bi-infinite extensions of *parabolic towers* developed by one of the authors in his study of geometric limits of rational maps [Ep]. This allows us to invoke the "soft" results of [Ep] to replace some of the most challenging parts of the usual renormalization theory.

The title of the paper refers to the attracting fixed point of the parabolic renormalization operator. The universal map, whose existence we prove, is a critical circle homeomorphism with a parabolic fixed point. The associated rotation number, which is of course 0, may be viewed as an infinite continued fraction, each entry of which equals to ∞.

Let us comment on the relation of this work to other open questions in holomorphic dynamics. The action of Shishikura's parabolic renormalization operator is a matter of considerable interest. The convergence (or divergence) of this procedure appears related to the measure-theoretic properties of quadratic Julia sets, one of the "holy grails" of the subject. Much of our approach translates to Shishikura's setting, although, some crucial ingredients are missing. Finally, the new definition of renormalization of circle maps given in [Ya3] was used there to show the hyperbolicity of the periodic orbits of renormalization, but the existence of these periodic points was proved in the classical context of commuting pairs. In the degenerate case studied in this paper, we handle both questions simultaneously. Perhaps this can give clues on how to do the same in the general case.

1. Preliminaries

Some notations. We use dist and diam to denote the Euclidean distance and diameter in $\mathbb{C}$. The notation $D_r(z)$ will stand for the Euclidean disk with center $z \in \mathbb{C}$ and radius r. The unit disk $D_1(0)$ will be denoted $\mathbb{D}$. By the circle $\mathbb{T}$ we understand the Euclidean manifold $\mathbb{R}/\mathbb{Z}$; it is naturally identified with the unit circle S^1 via $\theta \mapsto e^{2\pi i\theta}$. The real translation $x \mapsto x+\theta$ projects to the rigid rotation by angle θ, $R_\theta : \mathbb{T} \to \mathbb{T}$. For two points a and b in the circle $\mathbb{T}$ which are not diametrically opposite, $[a,b]$ will denote the shorter of the two arcs connecting them; as usual, $|[a,b]|$ will denote the length of this arc. For two points $a, b \in \mathbb{R}$, $[a,b]$ will denote the closed interval with endpoints a and b, regardless of their order. Unless otherwise specified, *the cylinder* will mean the Euclidean manifold $\mathbb{C}/\mathbb{Z}$. Its equator is the circle $\mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$. We denote π the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. We shall say that $U \subset \mathbb{C}/\mathbb{Z}$ is an *equatorial annulus* if U is a topological annulus with a smooth boundary and $U \supset \mathbb{T}$. Given a hyperbolic Riemann surface X, we shall denote by

$d_X(\cdot,\cdot)$ the Poincaré distance on X. We will sometimes use a symbol ∞ as a number with the conventions $1/\infty = 0$, $1+\infty = \infty$.

For a bounded domain $D \subset \mathbb{C}$ we will denote $\mathbf{B}(D)$ the Banach space of bounded analytic functions on D equipped with the sup norm. Similarly, for an equatorial annulus U we shall denote $\mathbf{A}(U)$ the space of bounded analytic maps $\phi : U \to \mathbb{C}/\mathbb{Z}$, such that $\phi|_{\mathbb{T}}$ is homotopic to Id, equipped with the uniform metric. We shall turn $\mathbf{A}(U)$ into a Banach manifold as follows. Denote $\tilde{U}$ the lift $\pi^{-1}(U) \subset \mathbb{C}$. The space of bounded functions $\tilde{\phi} : \tilde{U} \to \mathbb{C}$ which are analytic, and 1-periodic, so that $\tilde{\phi}(z+1) = \tilde{\phi}(z)$, becomes a Banach space when endowed with the sup norm. Denote this space by $P(U)$. For a function $\phi : U \to \mathbb{C}/\mathbb{Z}$ let $\check{\phi}$ be an arbitrarily chosen lift, so that $\pi(\check{\phi}(\pi^{-1}(z))) = \phi$. Observe that $\phi \in \mathbf{A}(U)$ if and only if $\tilde{\phi} = \check{\phi} - \mathrm{Id} \in P(U)$. We use the local homeomorphism between $P(U)$ and $\mathbf{A}(U)$ given by

$$\tilde{\phi} \mapsto \pi \circ (\tilde{\phi} + \mathrm{Id}) \circ \pi^{-1}$$

to define the atlas on $\mathbf{A}(U)$. The coordinate change transformations are given by $\tilde{\phi}(z) \mapsto \tilde{\phi}(z+n) + m$ for $n, m \in \mathbb{Z}$, therefore with this atlas $\mathbf{A}(U)$ is a Banach manifold.

We assume that the reader is familiar with the definition and basic properties of quasiconformal mappings. We use the notation σ_0 for the trivial Beltrami differential representing the standard complex structure in a planar domain.

Recall that a holomorphic mapping $f : X \to Y$ between Riemann surfaces is *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. The map f is a *branched covering*, if each $y \in Y$ has a neighborhood V such that $f : U \to V$ is proper for each component U of $f^{-1}(V)$. It is not hard to see that the former condition implies the latter.

Critical circle mappings. A critical circle map is an orientation preserving homeomorphism $\mathbb{T} \to \mathbb{T}$ of class C^3 with a single critical point c. We make a further assumption that the critical point is of cubic type. This means that for a lift $\bar{f} : \mathbb{R} \to \mathbb{R}$ of a critical circle map f with critical points at integer translates of $\bar{c}$, we have

$$\bar{f}(x) - \bar{f}(\bar{c}) = (x - \bar{c})^3(\mathrm{const} + O(x - \bar{c})).$$

We note that all the renormalization results will hold true if in the above definition "3" as the order of the critical point is replaced by any other odd number. We will always normalize a critical circle map f by placing the critical point of f at $0 \in \mathbb{T}$.

Being a homeomorphism of the circle, a critical circle map f has a well-defined rotation number, denoted $\rho(f) \in \mathbb{T}$. It is not difficult to show (see e.g. [MvS]) that when this number is irrational the map f is topologically semi-conjugate to the rigid rotation $R_{\rho(f)}$. A celebrated result of Yoccoz [Yoc] implies that the semi-conjugacy is, in fact, a true conjugacy. If x_0 is a periodic point of f with period q, and $x_0, x_1, \ldots, x_{q-1}$ is the orbit of x_0 listed in the counterclockwise order, then the combinatorial rotation number of x_0 is p/q where $f(x_0) = x_p$. Recall that $\rho(f)$ is a rational number equal to p/q in the reduced form if and only if f has a periodic point with combinatorial rotation number p/q.

We will find it useful to express the rotation number as a continued fraction

$$\rho(f) = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}} \tag{1.1}$$

(further on we will abbreviate such an expression as $[r_0, r_1, r_2, \dots]$ for typographical convenience.) If the rotation number is irrational, there is a unique infinite expansion (1.1) with $r_i \in \mathbb{N}$. For a rational rotation number, we will use an expansion $[r_0, \dots, r_n]$ with a finite number of terms, such that $r_i \in \mathbb{N}$ for $i < n$ and $r_n = \infty$. Each rational number has two expansions of this kind, $[r_0, \dots, r_{n-2}, k+1, \infty]$ and $[r_0, \dots, r_{n-2}, k, 1, \infty]$. We will use the dynamics of f to resolve this ambiguity below.

Let f be a critical circle map, and denote by $\tilde{f} : \mathbb{R} \to \mathbb{R}$ the lift of f with $\tilde{f}(0) \in [0, 1)$. Let us define a possibly finite sequence of positive rationals $\{p_m/q_m\}$ as follows. If there is a natural number m such that $\tilde{f}^{m+1}(0) - 1 \in [0, \tilde{f}(0))$ then $p_0/q_0 = 1/m$, otherwise, the sequence is empty. In the former case, let I_0 be the projection of the interval $[0, \tilde{f}(0)]$ onto the circle. Now inductively define p_m/q_m, and I_m as follows: let q_m be the smallest natural number for which $f^{q_m}(I_{m-1}) \ni 0$, and let $p_m \in \mathbb{N}$ be such that $\tilde{f}^{q_m}(\tilde{I}_{m-1}) - p_m \ni 0$ (here $\tilde{I}_{m-1}$ is the lift of I_{m-1} containing 0), and set $I_m \subset \mathbb{T}$ to be the closed arc between 0, $f^{q_m}(0)$ which does not contain $f^{q_{m-1}}(0)$. It is easy to see that when $\rho(f)$ is irrational, the numbers p_m/q_m are the finite convergents of the infinite continued fraction (1.1):

$$p_m/q_m = [r_0, r_1, \dots, r_{m-1}]. \tag{1.2}$$

On the other hand, if $\rho(f)$ is rational, the sequence $\{p_m/q_m\}$ is finite, and if n is the number of its terms, then the equations (1.2) uniquely define a finite continued fraction

$$\sigma(f) = [r_0, r_1, \dots, r_{n-1}, \infty]. \tag{1.3}$$

Of course, $\sigma(f)$ is a continued fraction expansion of $\rho(f)$, but now it is prescribed unambiguously. Since it reflects the ordering of the critical orbit of f with respect to the standard orientation of the circle, we will refer to $\sigma(f)$ as the *combinatorial type* of f. When the rotation number of f is irrational we will denote by $\sigma(f)$ the continued fraction (1.1), and again call it the combinatorial type of f. While a rotation number is invariant under a conjugacy which preserves the orientation, the combinatorial type is preserved if the conjugacy also fixes the marked point 0.

Note that we may still define the quantities $\rho(f)$ and $\sigma(f)$ in the case when f is a homeomorphism of a topological circle S with a marked point.

An important one-parameter family of examples of critical circle maps, the so-called standard (or Arnold's) maps, is obtained as follows. Given any $\theta \in \mathbb{R}$ the map

$$A_\theta(x) = x + \theta - \frac{1}{2\pi} \sin 2\pi x$$

commutes with the unit translation: $A(x+1) = A(x)+1$. Therefore it has a well-defined projection to an endomorphism of the circle. We denote this endomorphism by f_θ, although in reality it only depends on the class $\theta \bmod \mathbb{Z}$. An elementary computation shows that every A_θ is strictly monotone, with critical points, all of cubic type, at integer values of x. In particular, each f_θ is a critical circle mapping. Standard considerations imply that

$$\theta \to \rho(\theta) \equiv \rho(f_\theta)$$

is a continuous monotone map of $\mathbb{T}$ onto itself. Whenever $t \in \mathbb{T}$ is irrational, $\rho^{-1}(t)$ is a single point. For $t = p/q$ the set $\rho^{-1}(t)$ is a closed interval: for each parameter value in this interval the homeomorphism f_θ has a periodic orbit with combinatorial rotation number p/q, and this orbit has eigenvalue one at both endpoints θ_1, θ_2 of the interval. The combinatorial types of the maps f_{θ_1}, f_{θ_2} are different: if we assume that $\sigma(f_{\theta_1}) = [r_0, \ldots, r_{n-2}, k+1, \infty]$ then $\sigma(f_{\theta_1}) = [r_0, \ldots, r_{n-2}, k, 1, \infty]$. In particular, the interval $\rho^{-1}(0)$ is bounded by maps with parabolic fixed points whose combinatorial types are $[\infty]$ and $[1, \infty]$.
The significance of these examples in Renormalization Theory is due in part to the following completeness property of the family of Arnold's maps:

Proposition 1.1 (cf. [dF1, dF2])**.** *Every complex-analytic and real-symmetric endomorphism of $\mathbb{C}/\mathbb{Z}$ which is topologically conjugate to a member of the family $\{f_\theta\}$ itself belongs to this family.*

In combination with the properties of the dependence $\rho(\theta)$ quoted above, this yields a corollary:

Proposition 1.2 (Rigidity). *Let g be a complex-analytic and real-symmetric endomorphism of the cylinder $\mathbb{C}/\mathbb{Z}$. Suppose that g is conjugate by an orientation-preserving homeomorhism to a map f_θ such that either $\rho(\theta)$ is irrational, or f_θ has a periodic orbit with eigenvalue one in $\mathbb{T}$. Then $g \equiv f_\theta$.*

The above rigidity property will allow us to use Arnold's maps as quasiconformal models, and in this way obtain a straightening theorem for parabolic towers constructed in §5. There are other families of critical circle mappings which may be used for the same purpose. One notable example is the family of cubic Blaschke products

$$B_\theta(z) = e^{2\pi i\theta} z^2 \frac{z-3}{1-3z}.$$

The analogue of Proposition 1.2 is elementary for the family B_θ. We chose f_θ as our models, however, to be consistent with the existing literature on the subject.

2. Parabolic renormalization

A review of parabolic bifurcation. We present here a review of the theory of parabolic bifurcations, as applied in particular to an analytic critical circle mapping. A detailed discussion of parabolic bifurcation may be found in a paper of Shishikura [Sh]. The covering properties of the Écalle maps are studied, in particular, in one of the authors' graduate thesis [Ep]. That work is concerned with developing the Fatou-Sullivan structure theory for analytic mapping of finite type, of which the Écalle maps are a particular example. However, in our specific situation, less general arguments may be used, akin to those given in [BE].

Let f be an analytic map defined in some region $\mathrm{Dom}(f) \subset \mathbb{C}$. Assume that f has a parabolic periodic point p of period n with unit eigenvalue: $(f^n)'(p) = 1$. Writing

$$f(z) = (z-p) + a(z-p)^{n+1} + \text{higher order terms},$$

we say that a unit vector v specifies an *attracting direction* if $av^n \in \mathbb{R}_{<0}$. For each infinite orbit $\{f^{nk}(z)\}$ which converges to the parabolic point, there is one of the n vectors v with this property, for which the unit vectors

$$(f^{nk}(z)-p)/|f^{nk}(z)-p| \underset{k\to\infty}{\longrightarrow} v.$$

We say in this case that the orbit converges to p in the direction of v. For each v, we say that a topological disk U^A is an *attracting petal* of f at p if $\overline{U^A} \ni \{p\}$ and $f^n(\overline{U^A}) \subset U^A \cup \{p\}$, and an infinite orbit $\{f^{nk}(z)\}$ is eventually contained in U if and only if it converges to p in the direction of v. Similarly, U^R is a *repelling petal* for f if it is an attracting petal for the local branch of f^{-1} which fixes p. We further say that p is a *simple parabolic point* if $n = 1$, and thus there is a single attracting direction. Equivalently, any two attracting petals have a non-empty intersection. In this case a pair of petals U^A, U^R contains a punctured neighborhood of the parabolic point p.

Theorem 2.1 (Fatou Coordinates). *In the above notation,*

$$\bigcap_{k=0}^{\infty} f^{nk}(\overline{U^A}) = \{p\}, \text{ and } \bigcap_{k=0}^{\infty} f^{-nk}(\overline{U^R}) = \{p\}.$$

Moreover, there exist injective analytic maps

$$\phi^A : U^A \to \mathbb{C} \text{ and } \phi^R : U^R \to \mathbb{C},$$

unique up to post-composition by translations, such that

$$\phi^A(f^n(z)) = \phi^A(z) + 1 \text{ and } \phi^R(f^n(z)) = \phi^R(z) - 1.$$

The Riemann surfaces $C^A = U^A/f^n$ and $C^R = U^R/f^n$ are conformally equivalent to the cylinder $\mathbb{C}/\mathbb{Z}$.

By the Implicit Function Theorem, there exists a small enough neighborhood $W(f) \subset \mathbf{B}(\mathrm{Dom}(f))$ such that the collection of mappings with a parabolic orbit of period n forms a codimension one submanifold $\mathcal{W}$ in $W(f)$. We may simultaneously normalize the Fatou coordinates in $\mathcal{W}$ by selecting a pair of points $x \in U^A$, $y \in U^R$ for f and requiring that $\phi^A(x) = \phi^R(y) = 0$ both for f and for every other map $g \in \mathcal{W}$. With these normalizations, the Fatou coordinates depend locally continuously on the map throughout $\mathcal{W}$. Moreover:

Proposition 2.1. *Let G be a proper subdomain of U^A. Then the dependence $f \mapsto \phi^A$ extends to an analytic operator from a neighborhood of f in $\mathcal{W}$ to $\mathbf{B}(G)$, and similarly for the repelling Fatou coordinate.*

Proof. While this is not stated explicitly in [Sh], it follows from the argument on p. 259 of that paper in conjunction with the Ahlfors-Bers Theorem on the analytic dependence of the solution of a Beltrami equation on a parameter. □

We denote $\pi_A = \pi_{A,f} : U^A \to C^A$ and $\pi_R = \pi_{R,f} : U^R \to C^R$ the natural projections. The quotients C^A and C^R are customarily referred to as *Écalle–Voronin cylinders*; we will find it useful to regard these as Riemann spheres, with distinguished points $\pm$ filling in the corresponding punctures. Any conformal *transit homeomorphism* $\tau : C^A \to C^R$ fixing the ends $\pm$ is a translation in the standard cylinder coordinates. Lifiting it produces a map $\tilde{\tau} : U^A \to \mathbb{C}$ satisfying

$$\tau \circ \pi_A = \pi_R \circ \tilde{\tau}.$$

We will write $\tau \equiv \tau_\theta$, and $\tilde{\tau} = \tilde{\tau}_\theta$, where

$$\phi^R \circ \tilde{\tau} \circ (\phi^A)^{-1}(z) \equiv z + \theta \bmod \mathbb{Z}.$$

Let $z \in C^R$ and take an arbitrary preimage $\zeta = \pi_R^{-1}(z) \in U^R$. Suppose that the forward orbit of ζ intersects the attracting petal U^A, and denote by $n(\zeta)$ the first moment that $f^{n(\zeta)}(\zeta) \in U^A$. Define $E_f(z) = \pi_A(f^{n(\zeta)}(\zeta)) \in C^A$. Clearly this expression does not depend on the choice of the preimage ζ, and defines an analytic map from an open set in C^R to C^A, which we will call the *Écalle map.*

Of particular importance to us will be the case when f is an analytic critical circle map, defined by analytic extension in some annular neighborhood $\mathrm{Dom}(f) \subset \mathbb{C}/\mathbb{Z}$ of $\mathbb{T}$. In this case we may select $\mathbb{T}$-symmetric petals U^A and U^R. The intersections $\mathbb{T} \cap U^A$ and $\mathbb{T} \cap U^R$ will project to the *natural equators* of the corresponding Écalle-Voronin cylinders $E^A \subset C^A$ and $E^R \subset C^R$.

We now turn our attention to small perturbations of parabolic maps. For an analytic critical circle map g which is sufficiently close to f in the uniform metric on $\mathrm{Dom}(f)$, the parabolic point splits into a complex-conjugate pair of repelling fixed points $p_g \in \mathbb{H}/\mathbb{Z}$ and $\bar{p}_g$ with multipliers $e^{\pm 2\pi i \alpha(g)}$. It is easily seen from the Holomorphic Index Formula that if we select a a neighborhood $\mathcal{W}(f) \subset \mathbf{A}(\mathrm{Dom}(f))$ small enough, then for every critical circle map $g \in \mathcal{W}(f)$ we have $|\arg \alpha(g)| < \pi/4$. In this situation one may still speak of attracting and repelling petals:

Lemma 2.2 (Douady Coordinates). *For every critical circle map $g \in \mathcal{W}(f)$ there exist $\mathbb{T}$-symmetric topological discs U^A_g and U^R_g whose union is a neighborhood of p, and injective analytic maps*

$$\phi^A_g : U^A \to \mathbb{C} \text{ and } \phi^R_g : U^R_f \to \mathbb{C}$$

unique up to post-composition by translations, such that

$$\phi^A_g(g^n(z)) = \phi^A_g(z) + 1 \text{ and } \phi^R_g(g^n(z)) = \phi^R_g(z) - 1,$$

and such that the quotients $C_g^A = U_g^A/g^n$ *and* $C_g^R = U_g^R/g^n$ *are Riemann surfaces conformally equivalent to* $\mathbb{C}/\mathbb{Z}$.

Assigning the value at an arbitrarily selected basepoint specifies a Fatou or Douady coordinate uniquely. For further convenience the following choice will be made. Select $k, m \in \mathbb{N}$ large enough so that $a = g^k(0) \in U_g^A$ and $r = (g|_{\mathbb{T}})^{-m}(0) \in U_g^R$ throughout $\mathcal{W}(f)$, and set

$$\phi^A(f^k(0)) = \phi_g^A(g^k(0)) = 0, \text{ and } \phi^R((f|_{\mathbb{T}})^{-m}(0)) = \phi_g^R((g|_{\mathbb{T}})^{-m}(0)) = 0.$$

The following fundamental theorem first appeared in [DH]:

Theorem 2.2 (Continuity of Douady Coordinates). *With these normalizations we have*

$$\phi_g^A \to \phi^A \text{ and } \phi_g^R \to \phi^R$$

uniformly on compact subsets of U^A *and* U^R *respectively.*
Moreover, if $n(g) \in \mathbb{N}$ *is minimal with* $g^{n(g)}(a) \geq r$, *then*

$$g^{n(g)}(z) = (\phi_g^R)^{-1} \circ T_{\theta(g)+K} \circ \phi_g^A$$

wherever both sides are defined. In this formula $T_x(z)$ *denotes the translation* $z \mapsto z + x$, $\theta(g) \in [0,1)$ *is given by*

$$\theta(g) = 1/\alpha(g) + \underset{\alpha(g)\to\infty}{o(1)} \mod 1,$$

and the real constant K *is determined by our choice of the basepoints* a, r. *Thus for a sequence* $\{g_k\} \subset \mathcal{W}(f)$ *converging to* f, *the iterates* $g_k^{n(g_k)}$ *converge locally uniformly if and only if* $\theta(g_k) \to \theta$, *and the limit in this case is a certain lift of the transit homeomorphism* τ_θ *for the parabolic map* f.

Definition of parabolic renormalization of critical circle maps. Let f be an analytic critical circle map with a simple parabolic orbit of period n, and a distinguished orbit point $p = \lim f^n(0)$. Let $z \in C^R$ and take an arbitrary preimage $\zeta = \pi_R^{-1}(z) \in U^R$. Suppose that the forward orbit of ζ intersects the attracting petal U^A, and denote by $n(\zeta)$ the first moment that $f^{n(\zeta)}(\zeta) \in U^A$. Define $E_f(z) = \pi_A(f^{n(\zeta)}(\zeta)) \in C^A$. Clearly this expression does not depend on the choice of the preimage ζ, and defines an analytic map from a neighborhood of the equator of C^R to C^A. It is not difficult to see that the restriction $E_f|_{\mathbb{T}}$ is a critical circle map.
Let us recall briefly the definition of cylinder renormalization $\mathcal{R}_{\text{cyl}}$ given in [Ya3]. Suppose f is a critical circle mapping with an irrational rotation number, and p_1, p_2 a pair of complex conjugate repelling fixed points for the iterate f^{q_n}. Let l be a $\mathbb{T}$-symmetric simple arc connecting this points with the property that $f^{q_n}(l)$ is also a simple arc whose interior is disjoint from that of l. The bounded topological disk C bounded by l and $f^{q_n}(l)$ is a *fundamental crescent* of f if the iterate f^{q_n} is univalent on C, and the quotient

$$\bar{C} \cup f^{q_n}(C)/f^{q_n} \simeq \mathbb{C}/\mathbb{Z}.$$

Choosing a real-symmetric uniformizing coordinate on C, we see that the dynamical first return map of C projects to an analytic critical circle mapping g defined on a neighborhood W of $\mathbb{T}$. It is called a *cylinder renormalization* of f. To define the cylinder renormalization operator $\mathcal{R}_{\rm cyl}$, we select large enough n in the above procedure, so that the complex *a priori* bounds of [Ya1] imply that W contains an equatorial annulus of a fixed size, and set $\mathcal{R}_{\rm cyl}(f)=g$.

The continuity of Douady coordinates implies:

Proposition 2.3. *Suppose f_k is a sequence of Epstein critical circle maps with $f_k\to f$ and $\mathcal{R}_{\rm cyl}(f_k)\to g$. Assume further that f has a simple parabolic fixed point. Then*

$$g=\tau\circ E_f$$

where $\tau: C^A\to C^R$ is a transit isomorphism.

For any irrational number $\rho\in(0,1)$ the value $\theta=\theta(\rho)\in\mathbb{T}$ such that $\rho(\tau_{\theta(\rho)}\circ E_f)=\rho$ is unique, due to the standard considerations of monotonicity. If σ is the continued fraction expansion of ρ (1.1), we define the *σ-parabolic renormalization of f* as

$$\mathcal{P}_\sigma(f)=\tau_{\theta(\rho)}\circ E_f.$$

Fix a rational number $p/q\in[0,1)$, $(p,q)=1$, and let σ be a continued fraction expansion of p/q of the form (1.3). Select the smallest value $\theta=\theta(\sigma)\in\mathbb{T}$ for which the composition $\tau_\theta\circ E_f$ has a parabolic periodic orbit of combinatorial type σ. We call $\tau_{\theta(\sigma)}\circ E_f$ the *σ-parabolic renormalization of f* and write

$$\mathcal{P}_\sigma(f)=\tau_{\theta(\sigma)}\circ E_f.$$

In the case of $\sigma=[\infty]$, (which implies that the map f has a parabolic fixed point), we drop the index σ. We will see below that any such rational combinatorial type is realized by a unique value $\theta\in\mathbb{T}$ (Proposition 2.9).

In §8 we will extend the above definitions to some cases when f is an analytic map defined in a neighborhood W of an equatorial topological circle $S\subset\mathbb{C}/\mathbb{Z}$ with a single critical point $0\in S$ of cubic type. In the case when there is a simple parabolic orbit in S, we may still say that the composition $\tau_\theta\circ E_f$ is a parabolic renormalization of f if it has an invariant circle with a parabolic orbit of a given combinatorial type. However, selecting a transit map which realizes such a renormalization becomes problematic in this case. It will be resolved with the use of some global assumptions on the dynamics of f.

Covering properties of the Écalle maps. Let us begin by assuming that an analytic map $f: X\to Y$ between Riemann surfaces has a simple parabolic orbit of period n. Let $B\subset X$ be a component of the immediate basin of an element p of this orbit. The standard considerations imply:

Proposition 2.4. *Suppose B is a hyperbolic domain, and $f^n: B\to B$ a branched covering. Then B contains a critical point of f^n.*

Moreover,

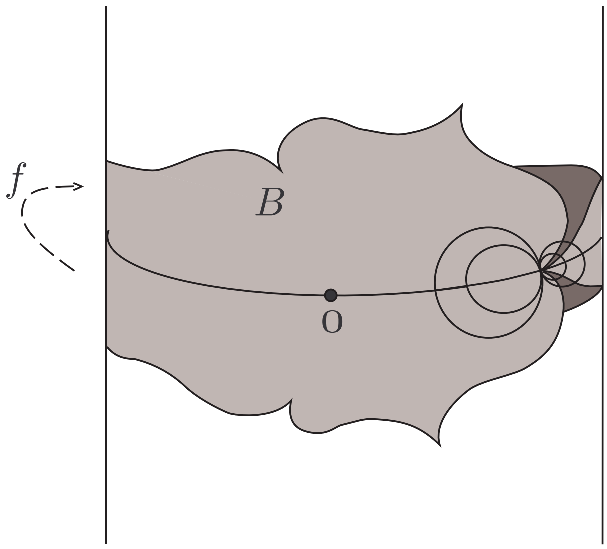


FIGURE 1. The proper immediate basin of a parabolic fixed point of a critical circle map

Proposition 2.5. *In addition to the assumptions of the above Proposition, suppose that the critical point c of $f^n : B \to B$ is unique, and has a local degree d. Then the attracting Fatou coordinate extends via the equation $\phi^A(f^n(z)) = \phi^A(z) + 1$ to an infinite-degree branched covering*

$$\phi^A : B \to \mathbb{C}.$$

It is branched at the points $z \in B$ for which $f^{nl}(z) = c$ for some $l \geq 0$, and all critical points of ϕ^A have the same degree d. (For further details, see [BE]*).*

We will say that a connected open set $B \subset \mathrm{Dom}(f)$ is a *proper immediate basin* of a parabolic point p of f if B is a connected component of the immediate basin of p such that $f^n(B) = B$, and the map $f^n : B \to B$ is proper. The degree of the basin is the degree of f^n as a branched covering of B. We illustrate this definition with the figure below, which depicts the proper immediate basin of a critical circle map with a simple parabolic fixed point. The crescents are two fundamental domains for the dynamics of f in the neighborhood of p, projecting to C^A and C^R under the Fatou coordinates. Note that the parabolic critical circle maps both in the family f_θ and in the family B_θ have proper immediate basins. The latter is clear, while the former follows from the analysis of the dynamics of f_θ which may be found, for example, in [dF2].
We now turn our attention to critical circle mappings. Firstly, we define a class of maps which will be a convenient setting to study parabolic renormalization. It contains, in particular, the maps in the Arnold's family f_σ, with a parabolic orbit in $\mathbb{T}$.

Definition 2.6. Let us denote $\mathbf{Par}_\sigma$ the collection of analytic critical circle maps f such that:

- f is defined in a $\mathbb{T}$-symmetric equatorial topological annulus $\mathrm{Dom}(f) \subset \mathbb{C}/\mathbb{Z}$ containing 0;
- f is a branched covering $\mathrm{Dom}(f) \to \mathbb{C}/\mathbb{Z}$ with a single critical value $f(0) \in \mathbb{T}$;
- finally, f has a simple parabolic periodic point $p \in \mathbb{T}$ of period n, whose immediate basin B is simply-connected, contains 0, and is proper and of degree 3.

The next two propositions will show that a parabolic renormalization of a map in $\mathbf{Par}_\sigma$ again possesses all of the above properties, and moreover, is a covering of a *finite* annulus over $\mathbb{C}/\mathbb{Z}$.

Proposition 2.7. *Suppose $f : \mathbb{T} \to \mathbb{T}$ is a critical circle mapping with a period n simple parabolic periodic point p in $\mathbb{T}$. Denoting by B the immediate basin of p, assume further that B is simply-connected, proper, of degree 3, and*

$$\mathbb{T} \setminus \cup_{j=0}^{n-1}\{f^j(p)\} \subset \cup_{j=0}^{n-1}(B).$$

Then the Écalle mapping $E_f : \mathrm{Dom}(E_f) \to \mathbb{C}/\mathbb{Z}$ is an infinite degree branched covering, ramified over a single critical value. Its domain $\mathrm{Dom}(E_f)$ is a bounded equatorial $\mathbb{T}$-symmetric annulus compactly contained in $\mathbb{C}/\mathbb{Z}$.

Proof. Let $B_1 = f^j(B)$ be the other component of the immediate basin of the parabolic orbit of f having the point p in its boundary. The annulus $\mathrm{Dom}(E_f)$ is the quotient $(B_1 \cap U^R)/f^n$ (see Fig. 2). The rest follows from Proposition 2.5. □

Proposition 2.8. *Let $f \in \mathbf{Par}_\sigma$, and $g = \mathcal{P}_{\sigma'}(f)$. Then $g \in \mathbf{Par}_{\sigma_1}$, and the annulus $\mathrm{Dom}(g)$ is bounded.*

Proof. The covering properties of f imply that every point on the circle which is not a part of the parabolic orbit, is contained in the parabolic basin. Thus, in view of the previous proposition, we only need to establish that the immediate basin of the parabolic orbit of g has the desired properties. Denote $p \in \mathbb{T}$ a parabolic point of g whose immediate basin B contains the critical point 0. Firstly, note that as g has a single critical value, there may only be one attracting direction at p, and hence, p is a simple parabolic point. Next, let us denote

$$\tilde{g} : \widetilde{\mathrm{Dom}(g)} \to \mathbb{C}$$

the lift of the map g via the projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Denoting c a critical value $g(0)$, we see that g is a degree three branched covering of a topological disk $A \ni 0$ over $(\mathbb{C} \setminus \mathbb{R}) \cup (c-1, c+1)$ (this restriction of g is, in fact, a map of the *Esptein class* (see [Ya1]). The domain A is compactly contained in the plane, and the properness of the basin B easily follows. □

Proposition 2.9. *Again, let $f \in \mathbf{Par}_\sigma$. Then for every rational or irrational combinatorial type σ' there exists a unique $\theta \in \mathbb{T}$ such that*

$$\sigma(\tau_\theta \circ E_f) = \sigma',$$

and if σ' is rational, then $\tau_\theta \circ E_f$ has a parabolic fixed point.

Proof. When σ is irrational, the claim follows from the standard considerations of monotone dependence on the parameter. Let us consider the case when σ is rational. A classical argument due to Fatou implies that since $E_f : \mathrm{Dom}(E_f) \to \mathbb{C}/\mathbb{Z}$ is a branched covering map, every non-repelling periodic orbit of $g_\theta \equiv \tau_\theta \circ E_f$ must attract a critical value of g_θ. Hence only one such orbit may exist. Now let g_{θ_0} have a parabolic orbit with combinatorial rotation number p/q, such that $\sigma(g_{\theta_0}) = \sigma$. Denoting $\tilde{g}_\theta$ the lift of g_θ with $\tilde{g}_\theta(0) \in [0,1)$, we have $\tilde{g}^q_{\theta_0}(x) = x + p$ for some values of $x \in \mathbb{R}$. Uniqueness of the non-repelling orbit of g_{θ_0} implies that the graph of $\tilde{g}_{\theta_0}$ must lie either entirely above, or entirely below the line $y = x + p$. Since the dependence $\theta \mapsto \tilde{g}_\theta(x)$ is strictly monotone, the values of θ_0 realizing each of these two possibilities are unique, and by our conventions they correspond to different combinatorial types representing the rotation number p/q. □

3. Statements of the results

For a finite continued fraction $\sigma_1 = [r_0, \ldots, r_n]$ with $r_m \in \mathbb{N} \cup \{\infty\}$ and a possibly infinite continued fraction $\sigma_2 = [s_0, \ldots, s_m, \ldots]$ with $s_m \in \mathbb{N} \cup \{\infty\}$ let us denote $\sigma_1 * \sigma_2$ the continued fraction $[r_0, \ldots, r_n, s_0, \ldots, s_m, \ldots]$. Let Σ^- be the collection of all formal continued fractions $[r_i]^0_{i=-\infty}$ where $r_i \in \mathbb{N} \cup \{\infty\}$, and the symbol ∞ appears infinitely often, and let Σ^+ be the set of all formal continued fractions $[r_i]^\infty_{i=0}$ with $r_i \in \mathbb{N} \cup \{\infty\}$. Finally, let Σ be the set of bi-infinite formal continued fractions $[r_i]^\infty_{i=-\infty}$ with $[r_i]^0_{i=-\infty} \in \Sigma^-$ and $[r_i]^\infty_{i=0} \in \Sigma^+$. Every continued fraction in Σ^-, can be factored as

$$[r_i]^0_{i=-\infty} = \cdots * \sigma_{-n} * \sigma_{-(n-1)} * \cdots * \sigma_0 \tag{3.1}$$

where a continued fraction σ_n is a finite fraction whose terms are natural numbers, except for the last one which is ∞ (such an expression represents a rational combinatorial type). Our main result is as follows:

Theorem 3.1. (Main Theorem) *There exists a collection $\mathcal{A}$ of analytic critical circle maps and a bijection $\iota : \Sigma^- \to \mathcal{A}$, which is continuous with respect to the coordinate topology on Σ^- and the uniform topology on $\mathcal{A}$, and has the following properties:*

(a) *the image $f_{[r_i]^0_{-\infty}} \equiv \iota([r_i]^0_{-\infty})$ belongs to $\mathbf{Par}_{\sigma_0}$, where σ_n is as in (3.1) and*

$$\mathcal{P}_{\sigma_1}(f_{[r_i]^0_{-\infty}}) = f_{\cdots * \sigma_{-n} * \sigma_{-(n-1)} * \cdots * \sigma_0 * \sigma_1}$$

(b) *for any σ, any $f \in \mathcal{P}_\sigma$, any $g \in \mathcal{A}$, and an arbitrarily chosen infinite sequence of rational combinatorial types $\sigma_1, \ldots, \sigma_n, \ldots$ we have*

$$\mathrm{dist}(\mathcal{P}_{\sigma_n} \circ \cdots \circ \mathcal{P}_{\sigma_1} f, \mathcal{P}_{\sigma_n} \circ \cdots \circ \mathcal{P}_{\sigma_1} g) \overset{i\to\infty}{\longrightarrow} 0$$

in the uniform metric on $\mathbb{T}$

(c) *suppose f is an analytic critical circle map with $\rho(f) = [a_0, a_1, \dots] \in \mathbb{R} \setminus \mathbb{Q}$, and assume that for some sequence $n_j \to \infty$ we have*

$$a_{n_j+i} \to r_i \text{ for all } i \leq 0, \text{ where } [r_i]_{-\infty}^0 \in \Sigma^-.$$

Then $\mathcal{R}_{\text{cyl}}^{n_j} f \to \iota([r_i]_{-\infty}^0)$.

(d) *finally, let $\bar{s} = [r_i]_{i=-\infty}^0 \in \Sigma^-$ be a periodic sequence with period k, $r_{i+k} = r_i$, and let f be the corresponding periodic point of the parabolic renormalization in $\mathcal{A}$. Suppose the finite formal continued fraction $[r_{-k+1}, \dots, r_0]$ factors into a product of combinatorial types $\sigma_m * \cdots * \sigma_1$, and denote*

$$P = \mathcal{P}_{\sigma_1} \circ \cdots \circ \mathcal{P}_{\sigma_m} : f \mapsto f.$$

Then there exists a neighborhood $U_f \subset \mathbb{T}$ in which f is defined, and a neighborhood Y of f in $\mathbf{A}(U_f)$ such that the following holds:
- *P extends to a compact complex-analytic operator $P : W \to W$, where W is a codimension 2 submanifold of Y, which contains the maps $f \in Y$ with $f(\bar{z}) = \bar{f}(z)$,*
- *the spectrum of the compact operator $D_f(P|_W)$ is compactly contained in the unit disk,*
- *moreover, for every $g \in \mathbf{Par}_{\sigma_1}$ there exists an iterate $P^n(g) \in W$, hence the rate of convergence $P^k g \to f$ is eventually geometric.*

Let us pause for a few remarks. The renormalization convergence statement in [Ya2], formulated for critical commuting pairs, is stronger than the statement (c) in the above theorem. However, the proofs of [Ya2] require different (and quite more involved) analytic techniques. Moreover, the proof of continuity of the mapping ι requires the tower rigidity result proved in [Ya2]. Otherwise, our exposition is self-contained. We note that we are unable to show that ι is a homeomorphism, although we expect this to be the case. The continuity of ι^{-1} would follow from the negative answer to the following question:

Do there exist rational combinatorial types σ and σ_1, an analytic cylinder-renormalizable critical circle map f, and a critical circle map $g \in \mathbf{Par}_\sigma$ with $\mathcal{R}_{\text{cyl}} f = \mathcal{P}_{\sigma_1} g$?

The corresponding statement for commuting pairs is quite obvious (see [Ya2]).

A corollary of the Main Theorem is the existence of the universal parabolic map announced in the title of this paper:

Theorem 3.2. (The Universal Parabolic Map) *There exists a critical circle map f_* with a parabolic fixed point in $\mathbb{T}$ with $\sigma(f) = [\infty]$ such that the following holds:*

(a) $\mathcal{P} f_* = f_*$

(b) *for any combinatorial type σ and any $f \in \mathbf{Par}_\sigma$ the sequence $\mathcal{P}^n f \to f_*$*

(c) *also for any analytic critical circle map f with $\rho(f) = [a_0, a_1, \dots]$ with $a_i \to \infty$, the renormalizations $\mathcal{R}_{\text{cyl}}^{\text{on}} f \to f_*$*

(d) *finally there exists a neighborhood* $U \supset \mathbb{T}$ *in which* f_* *is defined and a neighborhood* Y *of* f_* *in* $\mathbf{A}(U)$ *such that for a codimension* 2 *submanifold* $W \subset Y$ *which contains the critical circle maps in* Y, P *extends to an analytic operator* $W \to W$, *and* $D_{f_*}(\mathcal{P}|_Y)$ *is a compact operator whose spectral radius is less than* 1.

The rest of the paper is occupied by the proof of the Main Theorem. For the readers convenience let us outline the main steps of the argument here. In the following section we present an argument of C. Petersen which establishes a complex *a priori* bound on a parabolic renormalization of a map in $\mathbf{Par}_\sigma$. The compactness implied by such a bound is used in §5 to construct bi-infinite towers of renormalized maps, along the lines of [McM2]. A result of [Ep] is used to imply that a forward-infinite tower does not carry an invariant line field on its Julia set. In §6 we construct quasiconformal conjugacies between pinched polynomial-like restrictions of maps in $\mathbf{Par}_\sigma$. These conjugacies promote themselves to global quasiconformal conjugacies between bi-infinite renormalization towers. As a final stroke, in §7 we employ McMullen's hyperbolic expansion argument to show that a bi-infinite tower supports no invariant line fields, and hence is unique up to a homothety. The last section contains the proof of the Main Theorem based on the tower rigidity.

4. Petersen's complex *a priori* bounds

The following observation of Petersen [Pet] is of crucial importance:

Theorem 4.1 (Petersen). *Let* $f \in \mathbf{Par}_\sigma$, *and let* E_f *as before denote the Écalle mapping associated with one of the parabolic points of* f. *Then the modulus of the annulus* $\mathrm{Dom}(E_f)$ *is a universal positive constant, independent of* f.

Proof. The proof is illustrated in Fig. 2 for the case when f has a parabolic fixed point. As an excercise, the reader is invited to determine the combinatorial type of f in the figure.

Denote the rational rotation number of f by m/n. Note first, that the uniqueness of the critical value of f, and elementary considerations of the dynamics on the circle imply that every point in $\mathbb{T}$ except for the parabolic orbit itself is contained in its immediate basin. Let $p \in \mathbb{T}$ be the element of the parabolic orbit with $f^n(0) \to p$. Denote B_1, B_2 the two immediate parabolic basins of f whose boundaries contain p labelled so that $B_2 \ni 0$. In the case, when p is a fixed point, $B_1 = B_2 = B$. Let $\vartheta : B_2 \to \mathbb{D}$ be the real-analytic Riemann map specified uniquely by the requirement that 0 is mapped to 0. The map

$$b \equiv \vartheta \circ f^n \circ \vartheta^{-1} : \mathbb{D} \to \mathbb{D}$$

is a cubic Blaschke product. The local dynamical picture at p implies that one of the fixed points -1, 1 of b is parabolic, and the other is repelling. Assume without loss of generality that $b'(1) = 1$. There is a unique $\mathbb{T}$-symmetric Blaschke product of degree 3 satisfying the conditions $b(1) = 1$, $b'(1) = 1$, $b(-1) = -1$, $b'(0) = 0$, $b''(0) = 0$. In particular, the repelling eigenvalue $b'(-1) > 1$ does not depend on the choice of f. The annulus $\mathrm{Dom}(E_f)$ is precisely the quotient $(U^R \cap B_1)/f^n$ for any choice of repelling petal U^R at p, and is therefore isomorphic to $A = (D_r(-1) \cap \mathbb{D})/b$ for a sufficiently small positive r. The

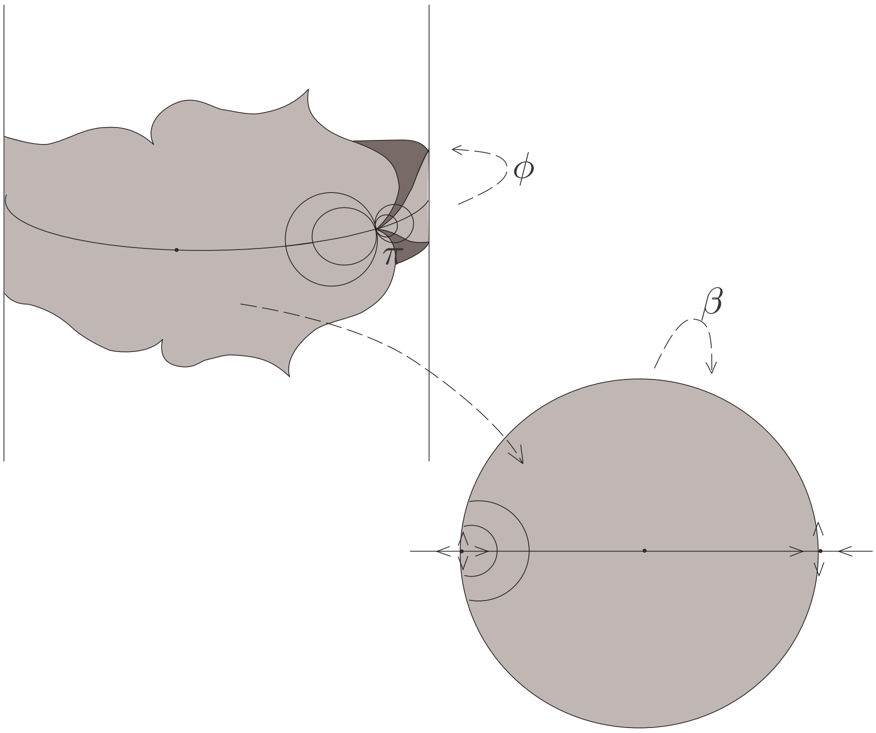


FIGURE 2. Petersen's argument

symmetry of the construction implies that A is a cylinder whose modulus is equal to half of the modulus of the quotient torus of -1, whence

$$\operatorname{mod}(\operatorname{Dom}(E_f)) = \operatorname{mod}(A) = \frac{\pi}{\log(b'(-1))}.$$

□

In fact, the following significantly stronger statement will be proved in §6 independently of the above result:

Theorem 4.2. *Let $\phi : \operatorname{Dom}(E_f) \to A_r$ denote the $\mathbb{T}$-symmetric conformal isomorphism onto a round annulus $A_r = \{|\operatorname{Im} z| < r\}/\mathbb{Z}$, normalized so that $\phi(0) = 0$. Then the composition*

$$\Psi = R_{-E_f(0)} \circ E_f \circ \phi^{-1}$$

does not depend on the choice of f (here $R_\theta : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ is, as before, the rotation by angle θ).

5. Towers of parabolic maps

Let $\mathcal{T} = (f_i)_{-n_1}^{n_2}$, $0 \le n_i \le \infty$ be a sequence of analytic critical circle maps. Assume that for each $i < n_2$ the map f_i has a simple parabolic orbit with a proper basin of degree

3. In the case when $n_2 < \infty$, we will require that $\rho(f_{n_2}) \in \mathbb{R} \setminus \mathbb{Q}$. We call $\mathcal{T}$ a *parabolic tower* if for every $i < n_2$

$$\mathcal{P}_{\sigma(f_{i+1})}(f_i) = f_{i+1}.$$

In the case when $n_1 = \infty$ we call $\mathcal{T}$ a *bi-infinite tower*, otherwise we say that $\mathcal{T}$ is a *forward-infinite tower*, and call f_{-n_1} the *base map of* $\mathcal{T}$. We use the word "infinite" even when $n_2 < \infty$, since in this case $\rho(f_{n_2}) \notin \mathbb{Q}$, hence f_{n_2} is infinitely renormalizable, and we may think of the successive renormalizations $\mathcal{R}^k_{\text{cyl}} f_{n_2}$ as the forward part of the tower. For a bi-infinite tower $\mathcal{T} = (f_i)^{n_2}_{-\infty}$ and $n < n_2$ let us denote $\mathcal{T}|_{\geq n}$ the forward-infinite tower $(f_i)^{n_2}_{n}$. The approach to renormalization convergence via rigidity of bi-infinite towers was pioneered by McMullen [McM2] in the context of Feigenbaum renormalization. Since his original proof, towers were utilized in similar contexts by others (see for example [dFdM2], and the work of one of the authors [Ya2]). Independently, the special case of forward-infinite parabolic towers considered in this paper was a central subject of the study by the other author [Ep] of geometric limits of finite-type complex-analytic maps. The present paper combines the use of the rigidity theorem of [Ep] and some of McMullen's arguments.

For two (formal) continued fractions $\sigma_1 = [r_1, \dots, r_k]$ and $\sigma_2 = [s_1, \dots]$ with terms in $\mathbb{N} \cup \{\infty\}$ let us denote $\sigma_1 * \sigma_2 = [r_1, \dots, r_k, s_1, \dots]$. Let us define the *combinatorial type* of a tower $\mathcal{T} = (f_i)^{n_2}_{-n_1}$ to be the formal continued fraction $\sigma(\mathcal{T})$ which in the case when $n_2 = \infty$ is the product $\cdots * \sigma(f_{-i}) * \cdots \sigma(f_0) * \cdots \sigma(f_i) * \cdots$, and when $n_2 < \infty$ is the product $\cdots * \sigma(f_{-i}) * \cdots \sigma(f_0) * \cdots * \sigma(f_{n_2-1}) * \sigma(f_{n_2})$. Thus, we have $\sigma(\mathcal{T}) \in \Sigma$ for any bi-infinite tower $\mathcal{T}$. A forward-infinite tower of any given combinatorial type may be assembled from the successive parabolic renormalizations of a given parabolic critical circle map. Let us show how examples of bi-infinite towers can be constructed. Consider a sequence of forward-infinite towers $\mathcal{T}_i = (f_{k,i})^{\infty}_{-m_i}$ with $m_i \to \infty$. By Petersen's complex *a priori* bounds, for every $k \in \mathbb{Z}$ the maps $f_{k,i}$ form a normal family in a neighborhood of the circle. We may therefore find a subsequence i_l for which

$$f_{k,i_l} \to g_k \text{ for all } k \in \mathbb{Z}.$$

Every g_k is a critical circle map. If we additionally assume that for each $k < n_2$ the denominators of the fractions $p_{k,i}/q_{k,i} = \rho(f_{k,i})$ form a bounded sequence, then every g_k has a rational rotation number p_k/q_k, and is necessarily parabolic. The considerations of continuity imply that $g_{k+1} = \mathcal{P}_{\sigma(g_k)} g_k$, whence the sequence $(g_k)^{n_2}_{-\infty}$ forms a parabolic tower. We will refer to such a tower as a *limiting bi-infinite tower*. As an excercise, we leave to the reader the proof of the following:

Proposition 5.1. *For any formal continued fraction $\bar{\sigma} \in \Sigma$ there exists a limiting bi-infinite tower $\mathcal{T}$ with $\sigma(\mathcal{T}) = \bar{\sigma}$.*

Consider a forward-infinite tower $\mathcal{T} = (f_i)^{n_2}_{-n_1}$. Since $\mathcal{P}_{\sigma(f_{i+1})}(f_i) = f_{i+1}$, we may naturally identify the domain of definition $\mathrm{Dom}(f_{i+1})$ with a subdomain of the repelling Fatou cylinder $C^R_{f_i}$. Denote by $\pi_{A,i} : U^A_{f_i} \to C^A_{f_i}$, and $\pi_{R,i} : U^R_{f_i} \to C^R_{f_i}$ the projections to the Fatou cylinders of f_i, and let $(\pi_{A,i})^{-1}$, $(\pi_{R,i})^{-1}$ be any choice of inverse branches. Let τ_{i+1} be the transit isomorphism from

$$\mathcal{P}_{\sigma(f_{i+1})}(f_i) = \tau_{i+1} \circ E_{f_i},$$

and g_{i+1} its lift $(\pi_{R,i})^{-1} \circ \tau_{i+1} \circ \pi_{A,i}$. By Proposition 2.5 the mapping g_{i+1} extends to a branched covering $B_i \to \mathbb{C}/\mathbb{Z}$ where B_i is a component of the immediate basin of f_i. Let $p_i^{\mathcal{T}}$ be the projection $\pi_{R,i-1} \circ \cdots \circ \pi_{R,-n_1+1} \circ \pi_{R,-n_1}$ defined on an appropriate subset of $\mathrm{Dom}(f_{-n_1})$, and set $\hat{f}_i = (p^{\mathcal{T}})_i^{-1} \circ f_i \circ p_i^{\mathcal{T}}$, and $\hat{g}_{i+1} = (p_i^{\mathcal{T}})^{-1} \circ g_{i+1} \circ p_i^{\mathcal{T}}$. By the *dynamics of the tower* $\mathcal{T}$ we will understand the abelian semigroup of analytic maps $D(\mathcal{T})$ generated by the collection of maps $\{\hat{f}_i, \hat{g}_{i+1}\}$ for all $-n_1 \leq i \leq n_2$. The *orbit* of a point $x \in \mathbb{C}/\mathbb{Z}$ under the tower $\mathcal{T}$ is the set $O_{\mathcal{T}}(x) = \cup h(x)$, where the union is over all $h \in D(\mathcal{T})$ for which $h(x)$ is defined. Clearly the definitions of $D(\mathcal{T})$ and $O_{\mathcal{T}}(x)$ do not depend on the choices we have made. A point $x \in \mathrm{Dom}(f_{-n_1})$ will be called *escaping* if $O_{\mathcal{T}}(x) \setminus \mathrm{Dom}(f_{-n_1}) \neq \emptyset$. The *filled Julia set* $K(\mathcal{T})$ is defined to be the collection of all non-escaping points in $\mathrm{Dom}(f_{-n_1})$, the *Julia set* $J(\mathcal{T})$ is its boundary. By virtue of its definition, $J(\mathcal{T})$ is totally invariant under $D(\mathcal{T})$.

Given a parabolic tower $\mathcal{T} = (f_i)_{-n_1}^{n_2}$, and any $n \geq -n_1$ we will refer to the collection of maps $D(\mathcal{T}|_{\geq n})$ *the elements of level* n of $\mathcal{T}$.

Consider two such towers $\mathcal{T}_1 = (f_i^1)_{-n_1}^{n_2}$, $\mathcal{T}_2 = (f_i^2)_{-n_1}^{n_2}$. For every $i < n_2$ and $j = 1, 2$ the maps f_i^j have unique parabolic orbits; denote by $\pi_{A,f_i^j} : U^A_{f_i^j} \to \mathbb{C}/\mathbb{Z}$, and $\pi_{R,f_i^j} : U^R_{f_i^j} \to \mathbb{C}/\mathbb{Z}$ the corresponding projections. We say that $\mathcal{T}_2 = (f_i^2)_{-n_1}^{n_2}$ are quasiconformally (resp. analytically, etc.) conjugate if there exists a sequence of homeomorphisms $h_i : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ of the appropriate regularity such that

$$h_i \circ f_i^1 \circ h_i^{-1} = f_i^2 \text{ and } \pi_{R,f_i^2} \circ h_i \circ (\pi_{R,f_i^1})^{-1} = h_{i+1}.$$

For forward-infinite towers this is equivalent to the existence of a single homeomorphism $h : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ which conjugates the corresponding elements of $D(\mathcal{T}_i)$.
Parts (a)-(c) of our Main Theorem will follow from the following combinatorial rigidity property of bi-infinite towers:

Theorem 5.1 (Rigidity of bi-infinite towers). *Any two bi-infinite parabolic towers of the same combinatorial type are identical.*

The discussion in [Ep] plays a key role in the proof of Theorem 5.1. We extract the following two properties of bi-infinite towers from [Ep]:

Theorem 5.2 ([Ep] **Structure of the Julia set of a forward tower**). *Let $\mathcal{T}$ be a forward-infinite parabolic tower with base map f, where $f = \mathcal{P}_\sigma g$ for some $g \in \mathbf{Par}_{\sigma(g)}$. Then the Julia set of $\mathcal{T}$ is a compact full set with empty interior, $J(\mathcal{T}) = K(\mathcal{T})$, and the repelling periodic orbits of elements of $D(\mathcal{T})$ form a dense subset in $J(\mathcal{T})$. Moreover, for any compact subset $C \subset J(\mathcal{T})$ and any open set U with $U \cap J(\mathcal{T}) \neq \emptyset$, there exists $h \in D(\mathcal{T})$ with $h(U \cap \mathrm{Dom}(h)) \supset C$.*

Theorem 5.3 ([Ep] **Rigidity of forward towers**). *Again let $\mathcal{T}$ be a forward-infinite parabolic tower with base map f, where $f = \mathcal{P}_\sigma g$ for some $g \in \mathbf{Par}_{\sigma(g)}$. Then there exist no line fields on $J(\mathcal{T})$ which are invariant under $D(\mathcal{T})$.*

It is relevant here that $f = \mathcal{P}_\sigma g$ (but not necessarily g itself) is a map of finite type, posessing a single critical value. Similiar statements for parabolic towers of holomorphic commuting pairs were proved in [Ya2]. Since holomorphic commuting pairs do not have natural domains of analyticity, different approach had to be used in that case.

6. Pinched cubic-like maps

We now introduce an appropriate adaptation of Douady-Hubbard's notion of polynomial-like mapping (see [DH2]). We say that a triple $F = (f, U, V)$ is a *pinched cubic-like map* if U and V are simply-connected open sets in $\mathbb{C}$, sch that $U \subset V$, $V \setminus U \neq \emptyset$, $\partial U \cap \partial V \neq \emptyset$, and $f : U \to V$ is a branched covering of degree 3. As usual, the filled Julia set of F is defined to be the set of non-escaping points:

$$K(F) = \{z \in U;\ f^n(z) \in U \text{ for } z \in \mathbb{N}\},$$

and the Julia set $J(F) = \partial K(F)$. In accordance with the established terminology, two pinched cubic-like maps $F = (f, U_f, V_f)$ and $G = (g, U_g, V_g)$ are *hybrid equivalent* if there exists a quasiconformal map (*the hybrid equivalence*) $\psi : V_f \to V_g$ such that

$$\psi \circ f \circ \psi^{-1} = g \text{ and } \bar{\partial}\psi|_{K(F)} = 0 \text{ a.e.}$$

The reason for the introduction of pinched cubic-like mappings is the following:

Proposition 6.1. *Let f be an analytic critical circle mapping, which extends to a branched covering of a $\mathbb{T}$-symmetric topological annulus* $\mathrm{Dom}(f)$ *over $\mathbb{C}/\mathbb{Z}$ ramified over a single critical value. Assume additionally, that f has a simple parabolic periodic orbit in $\mathbb{T}$, and set $\rho(f) = p/q$. Denote B_f the component of the immediate parabolic basin of f containing* 0 *and suppose that it is proper. Then there exists a pinched cubic-like map $F = (f^q, U_f, V_f)$ such that $V_f \supset U_f \supset B_f$, and $K(F) = \overline{B_f}$. The intersection $\partial V_f \cap \partial U_f$ consists of one (if $q = 1$) or two (otherwise) parabolic points in ∂B_f (see Fig. 6). We denote these points p_1 and p_2.*

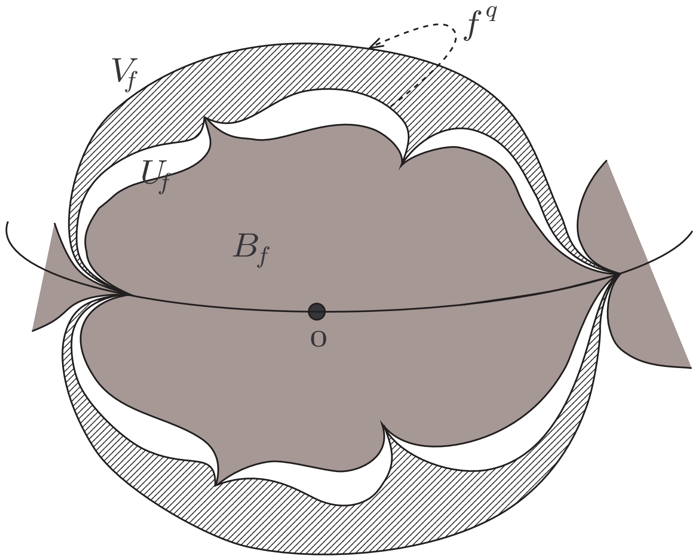
Proof. Denote by P_f the postcritical set of f, and observe that

$$f : \mathrm{Dom}(f) \setminus f^{-1}(P_f) \longrightarrow (\mathbb{C}/\mathbb{Z}) \setminus P_f$$

is an unbranched analytic covering map, hence a local isometry between the hyperbolic Riemann surfaces. Since the inclusion $\mathrm{Dom}(f) \setminus f^{-1}(P_f) \hookrightarrow (\mathbb{C}/\mathbb{Z}) \setminus P_f$ is strict, the Schwarz Lemma implies that the branches of f^{-1} contract the hyperbolic metric on $(\mathbb{C}/\mathbb{Z}) \setminus P_f$. For a small $k > 0$ denote γ_k the simple closed curve around B_f consisting of points lying at distance k from ∂B_f in the hyperbolic metric of $(\mathbb{C}/\mathbb{Z}) \setminus P_f$. Denote V the neighborhood of B_f bounded by γ_k, and U the connected component of $f^{-q}(V)$ containing 0. The contracting property of f^{-1} implies that $U \subset V$. By the same consideration, the filled Julia set of the triple $F = (f^q, U, V)$ is $\overline{B_f}$. Finally, if k is chosen small enough, the only points of the parabolic periodic orbit in $\bar{V}$ are p_1 and p_2. Since both of these points lie in P_f, and are the only such points in ∂B_f, the intersection $\partial V \cap \partial U = \{p_1, p_2\}$. □

We next show:

FIGURE 3. A pinched cubic-like restriction



Theorem 6.1. *Let $f \in \mathbf{Par}_{p/q}$, and $g \in \mathbf{Par}_{s/t}$. Again denote B_f (respectively B_g) the immediate basin of f (of g) containing 0. Then there exist pinched cubic-like maps $F = (f^q, U_f, V_f)$, $G = (g^t, U_g, V_g)$ such that the following holds:*

- *$V_f \setminus U_f$ is an annulus pinched at two parabolic points in ∂B_f, and similarly for $V_g \setminus U_g$;*
- *$K(F) = \overline{B_f}$ and $K(G) = \overline{B_g}$;*
- *F and G are quasiconformally equivalent; and moreover, when f and g are critical circle maps, F and G are hybrid equivalent.*

Proof. The general case will clearly follow if we establish the claim in the situation when f is an Arnold's map. At the first step of the proof we will present a pair of cubic-like maps F and G as above, and a quasiconformal map $\psi_0 : V_f \to V_g$ with the following properties:

- ψ_0 conjugates the action of $f^q|_{U_f}$ and $g^t|_{U_g}$ on their postcritical sets;
- $\psi_0(V_f \setminus U_f) = V_g \setminus U_g$;
- $\psi_0 \circ f^q \circ \psi_0^{-1}(z) = g^t(z)$ for $z \in \partial U_g$.

To do this we will slightly modify the construction of the domain of the cubic-like map given in the above Proposition.

Let us denote p_f, p'_f the elements of the parabolic orbit in ∂B_f. To fix our ideas, let B_f be the basin of attraction of p_f under the iterate f^q. Similarly label p_g, p'_g in the boundary of B_g. We shall begin the construction of the domains of the cubic-like maps and the hybrid equivalence $\psi : V_f \to V_g$ near p_f, p_g.

Let U^A_f, U^A_g be a pair of attracting petals of the parabolic points p_f, p_g respectively, and ϕ^A_f, ϕ^A_g the corresponding attracting Fatou coordinates. Let us select the repelling petals U^R_f, U^R_g so that in the attracting Fatou coordinates their boundaries form horizontal lines $\{\operatorname{Im} z = \pm N\}$ for some large value of N. Denote $\phi^R_f : U^R_f \to \mathbb{C}$ the repelling Fatou coordinate normalized so that $\{\operatorname{Re} z = 0\} \subset \phi^R_f(U^R_f)$, and define ϕ^R_g in a similar fashion. The images $\phi^R_f(\partial B_f)$, $\phi^R_g(\partial B_g)$ are two curves unrolling horizontally towards $-\infty$, invariant under $z \mapsto z+1$. Select $M > 0$ so that the strip $\{|\operatorname{Im} z| < M\}$ does not intersect $\phi^R_f(B_f) \cup \phi^R_g(B_g)$. Let $f(x)$ be a strictly decreasing smooth function with $f(0) > 0$ and $\lim_{x\to-\infty} f(x) = M$. Denote $L^\pm$ the curves

$$L^\pm = \{\operatorname{Im} z = \pm f(\operatorname{Re} z),\ \operatorname{Re} z \leq 0\},$$

and set $l^\pm_f = (\phi^R_f)^{-1}(L^\pm)$, $l^\pm_g = (\phi^R_g)^{-1}(L^\pm)$. By construction, l^+_f and l^-_f are two smooth arcs meeting at the parabolic point p_f, whose preimages by the branch of f^{-q} fixing p_f lie closer to ∂B_f, and similarly for $l^\pm_g$. These curves will become parts of the boundaries of the domains of the two cubic-like maps. Before completing the constructions of these domains, let us explain how the quasiconformal map ψ is constructed in a small neighborhood of p_f. Note that the impression of the postcritical set of f and g on the attracting Fatou cylinders is a single point $\hat{P}_f = \phi^A(P_f \cap U^A_f)/\mathbb{Z}$, and similarly for $\hat{P}_g$. Let us define $\psi_0 : U^R_f \to U^R_g$ as $(\phi^R_g|_{\phi^R_g(U_g)})^{-1} \circ \phi^R_f$. Let us now select an arbitrary quasiconformal map

$$H : \phi^A_f(U^A_f \setminus U^R_f)/\mathbb{Z} \to \phi^A_g(U^A_g \setminus U^R_g)/\mathbb{Z}$$

which maps $\hat{P}_f$ to $\hat{P}_g$ and coincides with $\phi^A_g \circ \psi_0 \circ (\phi^A_f)^{-1} \operatorname{mod} \mathbb{Z}$ on the boundary of the domain of definition. The map H has a lift $U^A_f \setminus U^R_f \to U^A_g \setminus U^R_g$ which coincides with ψ_0 on $\partial U^R_f \cap U^R_f$. Together the two mappings give us a desired quasiconformal map from a neighborhood of p_f to a neighborhood of p_g which conjugates the dynamics of f^q and g^t on the postcritical sets and maps $l^\pm_f \to l^\pm_g$. Having made clear the idea of the construction, we leave it to the reader to construct the boundary of the domain V_f and the map ψ_0 near p'_f. To complete the construction of V_f outside the neighborhoods of p_f, p'_f, let us set it equal to a smooth curve γ'_k lying between γ_k and γ_{k+1} (see the proof of the preceding Proposition), and similarly for V_g. Extend ψ_0 to a quasiconformal homeomorphism $h : V_f \to V_g$ which is a conjugacy on the postcritical sets in an arbitrary fashion. Let U_f and U_g be the connected components of $f^{-q}(V_f)$, $g^{-t}(V_g)$ respectively, containing 0, and define ψ_0 as the lift of h via the commuting diagram:

$$\begin{array}{ccc} U_f & \xrightarrow{\psi_0} & U_g \\ \Big\downarrow{\scriptstyle f^q} & & \Big\downarrow{\scriptstyle g^t} \\ V_f & \xrightarrow{h} & V_g \end{array}$$

(note that this definition agrees with the definition of ψ_0 on $U^R_f \cup U^A_f$). As an excercise, the reader is invited to verify that the map ψ_0 may be extended to a homeomorphism $V_f \to V_g$ which agrees with h on ∂V_f (this only needs to be checked near the first preimages of p_f

and p'_f). We have thus obtained the cubic-like maps $F = (f^q, U_f, V_f)$ and $G = (g^t, U_g, V_g)$ and the quasiconformal map ψ_0 with the desired properties. The Schwarz Lemma again imples that $K(F) = \overline{B_f}$ and $K(G) = \overline{B_g}$.

We may now utilize the standard pull-back argument to obtain a quasiconformal conjugacy between F and G. Inductively define $\psi_n : U_f \to U_g$ as a lift of ψ_{n-1} via

$$\begin{array}{ccc} U_f & \xrightarrow{\psi_n} & U_g \\ \Big\downarrow f^q & & \Big\downarrow g^t \\ V_f & \xrightarrow{\psi_{n-1}} & V_g \end{array}$$

and extend it by setting $\psi_n \equiv \psi_1$ in $V_f \setminus U_f$. Since f and g are analytic, all the maps ψ_n have the same quasiconformal dilatation bound, and we may select a locally uniformly converging subsequence $\psi_{n_k} \to \psi_\infty$. The limiting quasiconformal map is a conjugacy between the pinched cubic-like maps, however it may happen that $\mu = \psi_\infty^* \sigma_0$ does not vanish at almost every point. When G is real-analytic this can be rectified as follows. We define a Beltrami differential $\nu(z)$ on $\mathbb{C}/\mathbb{Z}$ by setting $\nu(z) = (f^n)^* \mu(z)$ for all z such that $f^n(z) \in B_f$, and $\nu(z) = \sigma_0$ elsewhere. By virtue of the definition, $||\nu||_\infty = ||\mu||_\infty$ and $f^* \nu = \nu$. By the Measurable Riemann Mapping Theorem, there exists a quasiconformal mapping $\phi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that $\phi^* \sigma_0 = \nu$. The invariance property of ν implies that $\phi \circ f \circ \phi^{-1}$ is an analytic map, which by Proposition 1.2 is equal to f. As the preimages of $\bar{B}_f$ are dense in ∂B_f, $\phi|_{\partial B_f} = \mathrm{Id}$. We define

$$\psi(z) = \begin{cases} \psi_\infty(z) \text{ for } z \notin B_f \\ \psi_\infty \circ \phi^{-1}(z) \text{ for } z \in B_f \end{cases}$$

By the Bers Sewing Lemma, $\psi : V_f \to V_g$ is quasiconformal, and hence a desired hybrid equivalence. □

We are now equipped to give the proof of Theorem 4.2 announced previously.

Proof of Theorem 4.2. Let f and g be two maps satisfying our assumptions, and $\psi : V_f \to V_g$ a hybrid equivalence given by the preceding theorem. Denote $\pi_{A,f} : U_f^A \to \mathbb{C}/\mathbb{Z}$, $\pi_{R,f} : U_f^R \to \mathbb{C}/\mathbb{Z}$, $\pi_{A,g} : U_g^A \to \mathbb{C}/\mathbb{Z}$, and $\pi_{A,g} : U_g^A \to \mathbb{C}/\mathbb{Z}$ the corresponding projections, normalized, as before, so that $E_f(0) = E_g(0) = 0$. Since $\psi(B_f) = B_g$ it induces two $\mathbb{T}$-symmetric analytic isomorphisms

$$\psi_R = \pi_{R_g} \circ \psi \circ \pi_{R_f}^{-1} : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g), \text{ and } \psi_A = \pi_{A_g} \circ \psi \circ \pi_{A_f}^{-1} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}.$$

The second map fixes 0, hence $\psi_A \equiv \mathrm{Id}$. As $\phi \circ f \circ \phi^{-1} = g$, we have

$$E_g \circ \phi_R \equiv \psi_A \circ E_f \equiv E_f.$$

The claim of the theorem follows from the uniqueness of the Riemann mapping. □

Theorem 6.2. *For every $\mu > 0$ there exists $K = K(\mu) > 1$ such that the following holds. Suppose that $f : \mathrm{Dom}(f) \to \mathbb{C}/\mathbb{Z}$ and $g : \mathrm{Dom}(g) \to \mathbb{C}/\mathbb{Z}$ satisfy the assumptions of Theorem 6.1, the annuli* $\mathrm{Dom}(f)$, $\mathrm{Dom}(g)$ *are bounded, and* $\mu = \min(\mathrm{mod}(\mathrm{Dom}(f)), \mathrm{mod}(\mathrm{Dom}(g)))$.

Then for every rational combinatorial type σ, there exists a $\mathbb{T}$-symmetric K-quasiconformal homeomorphism $\phi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that

$$\phi \circ \mathcal{P}_\sigma(f) \circ \phi^{-1} = \mathcal{P}_\sigma(g).$$

We will be most interested in the situation when f and g are two parabolic renormalizations of maps with proper basins. In this case the existense of a universal μ is guaranteed by the Petersen's *a priori* bound.

Proof. By the Grötzsch Inequality, there is an equatorial annulus $U = U(\mu) \supset \mathbb{T}$ in which both f and g are defined. By real *a priori* bounds there exists a K_1-quasisymmetric conjugacy $h : \mathbb{T} \to \mathbb{T}$, $h \circ \mathcal{P}_\sigma(f) \circ h^{-1} = \mathcal{P}_\sigma(g)$. Since critical circle maps in $\mathbf{A}(U)$ form a normal family, the constant K_1 may be chosen depending only on μ. Let ψ be the hybrid conjugacy from Theorem 6.1, and as before let an analytic homeomorphism $\psi_R : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g)$ fixing 0 be given by $\pi_{R_g} \circ \psi \circ \pi_{R_f}^{-1}$. By its definition, the map ψ_R extends to a quasiconformal map of a larger annulus $\pi_{R,f}(V_f) \to \pi_{R,g}(V_g)$. Denote ϕ_0 a quasiconformal homeomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that $\phi_0|_{\mathbb{T}} \equiv h$ and $\phi_0 \equiv \psi$ on an open neighborhood of $\partial\,\mathrm{Dom}(E_f)$. Again by a normal family argument, we may choose ϕ_0 with a dilatation bound depending only on μ; denote this bound $K = K(\mu)$. Now inductively define $\phi_i : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g)$ as a lift of ϕ_{i-1} via the commutative diagram

$$\begin{array}{ccc}
\mathrm{Dom}(E_f) & \xrightarrow{\phi_i} & \mathrm{Dom}(E_g) \\
\big\downarrow{\scriptstyle \mathcal{P}_\sigma(f)} & & \big\downarrow{\scriptstyle \mathcal{P}_\sigma(g)} \\
\mathbb{C}/\mathbb{Z} & \xrightarrow{\phi_{i-1}} & \mathbb{C}/\mathbb{Z}
\end{array}$$

and set $\phi_i \equiv \phi_0$ outside $\mathrm{Dom}(E_f)$. Clearly all maps ϕ_i have the same quasiconformal dilatation. Thus we may select a subsequence $\phi_{i_k} \to \phi$ uniformly on compact sets. By construction, ϕ is the desired conjugacy. □

We conclude this section with a step towards proving Theorem 5.1:

Theorem 6.3. *Any two bi-infinite towers of the same combinatorial type are quasiconformally conjugate.*

Proof. Let $\mathcal{T}_1 = (f_i)_{-\infty}^{\infty}$, $\mathcal{T}_2 = (g_i)_{-\infty}^{\infty}$ be two bi-infinite towers with $c(\mathcal{T}_1) = c(\mathcal{T}_2)$. As follows from Theorem 6.2 for every $i \in \mathbb{Z}$ there exists a K-quasiconformal map $h_i : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ with $h_i \circ f_i \circ h_i^{-1} = g_i$. By a pull-back argument, for every $j > i$ the map h_i induces a conjugacy $h_{i,j} \circ f_j \circ h_{i,j}^{-1} = g_j$. Now select a subsequence h_{i_k} such that $h_{i_k,j} \to \hat{h}_j$ uniformly on compact subsets of $\mathbb{C}/\mathbb{Z}$ for every $j \in \mathbb{Z}$. By definition, the sequence $\{\hat{h}_j\}$ is the desired conjugacy. □

7. McMullen's hyperbolic expansion argument

In this section we conclude the proof of Theorem 5.1. We utilize the argument developed by McMullen [McM2] in the context of quadratic-like maps to rule out the existence of non-trivial quasiconformal conjugacies between bi-infinite towers:

Theorem 7.1. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two bi-infinite towers, and let $h = (h_i)$ be the quasiconformal conjugacy from Theorem 6.3. Then for every i, $(\partial h_i/\partial\bar{z})/(\partial h/\partial z) = 0$ almost everywhere in $\mathbb{C}/\mathbb{Z}$.*

We will require some preliminary definitions. For a differentiable map $f : X \to Y$ of hyperbolic Riemann surfaces, $||f'(x)||_{X,Y}$ will stand for the norm of the derivative with respect to the hyperbolic metrics; we will simply write $||f'(x)||$ if $X = Y = (\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$. For the proofs of the next two lemmas see [McM2]:

Lemma 7.1. *There exists a continuous and increasing function $C(s) < 1$ with $C(s) \to 0$ as $s \to 0$ such that for the inclusion $\iota_{X,Y}$ of a hyperbolic Riemann surface X into a hyperbolic Riemann surface Y,*

$$||\iota'_{X,Y}(z)||_{X,Y} < C(s),$$

where $s = d_Y(x, Y \setminus X)$.

Lemma 7.2. *Let $f : X \to Y$ be an analytic map between hyperbolic Riemann surfaces with nowhere vanishing derivative. Suppose, in addition, that $X \subset Y$. Then for $z_1, z_2 \in X$ we have*

$$||f'(z_1)||_{Y,Y}^{1/\alpha} \leq ||f'(z_2)||_{Y,Y} \leq ||f'(z_1)||_{Y,Y}^{\alpha},$$

where $\alpha = exp(K d_Y(z_1, z_2))$ with a universal constant $K > 0$.

A consequence of the above expansion principles is the following:

Proposition 7.3. *Let $\mathcal{T}_{-\infty}^{n_2}$ be a bi-infinite tower, set $\mathcal{T}_n = \mathcal{T}|_{\geq n}$ and let $h \in D(\mathcal{T}_n)$. We have the following:*

(I) *$||h'(z)|| \geq 1$ for any $z \in \mathrm{Dom}(h)$, and $h(z) \notin \mathbb{T}$;*

(II) *there exists a universal constant $C > 1$ such that if $z \in \mathrm{Dom}(f_n)$ and $h(z) \in (\mathbb{C}/\mathbb{Z}) \setminus \mathrm{Dom}(f_n)$ then $||h'(z)|| > C$;*

(III) *moreover, suppose z and $h(z)$ is as in (II), and $k < n$. Then denoting $\zeta = (p_n^{\mathcal{T}|_{\geq k}})^{-1}(z)$, $\hat{h} = (p_n^{\mathcal{T}|_{\geq k}})^{-1} \circ h \circ p_n^{\mathcal{T}|_{\geq k}}$ for some branch of $(p_n^{\mathcal{T}|_{\geq k}})^{-1}$ we have $||\hat{h}'(\zeta)|| > C$.*

Proof. Fix $n \in \mathbb{Z}$. Select a sequence $\theta_k \in \mathbb{T}$ such that if we denote $g_k = R_{\theta_k} \circ f_n$, then g_k is a critical circle mapping with an irrational rotation number $\rho(g_k) = [r_0^k, r_1^k, \ldots, r_i^k, \ldots]$ with $r_i^k \xrightarrow{k\to\infty} \infty$. By Theorem 2.2 for every $h \in D(\mathcal{T}_n)$ there exists a sequence of iterates m_k such that $g_k^{m_k} \to h$ uniformly on compact subsets of $\mathrm{Dom}(h)$. By definition, every g_k is a branched covering $\mathrm{Dom}(f_n) \to \mathbb{C}/\mathbb{Z}$ with a single critical value $R_{\theta_k}(f_n(0))$ which is contained in $\mathbb{T}$. The map

$$g_k : X_k \equiv \mathrm{Dom}(f_n) \setminus (g_k^{-1}(\mathbb{T})) \to Y \equiv (\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$$

is unbranched, and hence a local isometry between the hyperbolic Riemann surfaces. By the Chain Rule, $||g_k'(z)|| = ||g_k'(z)||_{X_k,Y}/||\iota'_{X_k,Y}(z)||$. By Lemma 7.1, $||g_k'(z)|| > 1$ and the first claim follows.

Now suppose $z \in \mathrm{Dom}(f_n)$ is such that $f_n(z) \notin \mathrm{Dom}(f_{-n})$. By Petersen's complex *a priori* bounds, there exists a universal annulus $U \supset \mathbb{T}$ such that $g_k \in \mathbf{A}(U)$. Since $\mathbb{T}$ is invariant under g_k, normal family considerations imply that there exists another universal annulus $V \Subset U$ around $\mathbb{T}$ such that $g_k(x) \in \mathrm{Dom}(f_n)$ for $x \in V$. On the other hand, the same considerations imply that the preimage $g_k^{-1}(\mathbb{T}) \setminus V$ is a bounded Euclidean distance away from the point z. Hence there exists a universal constant $d > 0$ such that $\mathrm{dist}_Y(z, \partial X_k) \le d$. The second claim follows from Lemma 7.1 and the Chain Rule.

The third claim follows easily from the second and the Koebe Distortion Principle. $\Box$

Theorem 7.2. *Let $\mathcal{T}$ be a bi-infinite tower, let $z = z_i \in \mathrm{Dom}(f_i)$. For $n \le i$ let $z_n \in (p_i^{\mathcal{T}|_{\ge n}})^{-1}(z)$. Then* $\mathrm{dist}(z_n, J(\mathcal{T}|_{\ge n})) \underset{n\to-\infty}{\longrightarrow} 0$.

Proof. Assume that $z_n \notin J(\mathcal{T}|_{\ge n})$. Then for every k between i and $n-1$ there exists an element $h_k \in D(\mathcal{T}|_{\ge k})$ such that $h_k(z_k) \notin \mathrm{Dom}(f_k)$. Denote $\zeta_k^n = (p_k^{\mathcal{T}|_{\ge n}})^{-1}(h_k(z_k)) \in \mathrm{Dom}(f_n)$, and let $\psi_{k,n} \in D(\mathcal{T}|_{\ge n})$ be the element of level n mapping ζ_{k-1}^n to ζ_k^n. By Proposition 7.3 (III) we have $||\psi'_{k,n}(\zeta_{k-1}^n)|| > C$. By a normality argument based on Petersen's *a priori* bounds there exists a universal $s > 0$ such that $\mathrm{dist}_{(\mathbb{C}/\mathbb{Z})\setminus\mathbb{R}}(\zeta_i^n, J(\mathcal{T}|_{\ge n})) < s$. Denote α_n the hyperbolic geodesic of length $l(\alpha_n) < s$ connecting $h_n(z_n)$ to $J(\mathcal{T}|_{\ge n})$. Let α_n' be the connected component of the preimage $\psi_{i,n}^{-1}\circ\psi_{i+1,n}^{-1}\circ\cdots\circ\psi_{n-1,n}^{-1}(\alpha_n)$ containing ζ_i^n. Note that $\mathrm{dist}(\zeta_i^n, J(\mathcal{T}|_{\ge n})) \le l(\alpha_k')$. As we have seen, $||D\psi_{i,n}^{-1}\circ\psi_{i+1,n}^{-1}\circ\cdots\circ\psi_{n-1,n}^{-1}(h_n(z_n))|| > C^{n-i}$. By Lemma 7.2 this inequality holds along α_n' with C replaced by $C_1(C, s)$, and hence $l(\alpha_n') \to 0$. $\Box$

Proof of Theorem 7.1. Suppose that for some i the Beltrami differential $\mu_i(z) = (\partial h_i/\partial\bar{z})/(\partial h\partial z)$ is supported on a set of positive measure. Let $u_i(z) = \mu_i(z)/|\mu_i(z)|$ denote the corresponding line field. By Theorem 5.3 we may select a point z of almost continuity of $u(z)$ such that $z_n = (p_i^{\mathcal{T}|_{\ge n}})^{-1}(z) \notin J(\mathcal{T}|_{\ge n})$ for every $n \le i$. Let ζ_k^n, $\psi_{k,n}$ be as above. By normality considerations based on Petersen's complex *a priori* bounds we may find a subsequence $n_k \to -\infty$ such that $f_{j+n_k} \to g_j$ uniformly on compact subsets of the domain, $\zeta_{n_k}^{n_k} \to \zeta$ and $u_{j+n_k} \to u_j'$ in w^*-sense. Denote $\mathcal{T}'$ the tower $(g_j)_{-\infty}^{\infty}$.

Let D be a small disk around ζ in $(\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$. Denote D_{n_k} its univalent preimage around z_{n_k} by an inverse branch $h_{n_k}^{-1}$ of an element of $D(\mathcal{T}|_{\ge n_k})$ and $D'_{n_k} = p_i^{\mathcal{T}|_{\ge n}}(D_{n_k}) \ni z$. Proposition 7.3 together with Lemma 7.2 implies that the Euclidean diameters of D'_{n_k} shrink to 0, and z is well inside D'_{n_k}. The latter implies that the univalent maps $h_{n_k}\circ p_i^{\mathcal{T}|_{\ge n}}$ form a normal family in a neighborhood of z. As the line field u_i is almost constant in D'_{n_k}, we see that the line field u_0' is univalent in an open neighborhood D of ζ.

By Theorem 5.2 there exist elements $h_1, h_2 \in D(\mathcal{T}'|_{\leq 0})$ mapping D onto 0 and $g_0^2(0)$. By invariance, this means that u' is locally univalent at 0 and $g_0^2(0)$, which implies a contradictory behaviour at $g_0(0)$. $\square$

8. Proof of the Main Theorem

Parts (a) - (c).

Construction and properties of the mapping ι. Firstly let construct the mapping ι, and the set $\mathcal{A}$ as its image $\mathcal{A} = \iota(\Sigma^-)$. Let $[r_i]_{i=-\infty}^0 \in \Sigma^-$. Continue this formal continued fraction in an arbitrary fashion to a bi-infinite continued fraction $\bar{\sigma} \in \Sigma$. By Proposition 5.1 there exists a bi-infinite tower $\mathcal{T}_{\bar{\sigma}} = (f_i^{\bar{\sigma}})_{i=-\infty}^{n_2}$ with $\sigma(\mathcal{T}_{\bar{\sigma}}) = \bar{\sigma}$. We set

$$\iota([r_i]_{-\infty}^0) = f_0^{\bar{\sigma}}.$$

Let us see that this definition is independent of the choice of the forward part of $\bar{\sigma}$. Indeed, suppose $\bar{\sigma}' = [s_i]_{-\infty}^{\infty}$ with $s_i = r_i$ for $i \leq 0$, and $\mathcal{T}_{\bar{\sigma}'} = (f_i^{\bar{\sigma}'})_{i=-\infty}^{n_2'}$ is a bi-infinite tower with $\sigma(\mathcal{T}_{\bar{\sigma}'}) = \bar{\sigma}'$. Let us factor the forward part of $\bar{\sigma}$ into the product of combinatorial types $\sigma_1 * \sigma_2 * \cdots$. Set $h_0 = f_0^{\bar{\sigma}'}$ and inductively define $h_i = \mathcal{P}_{\sigma_i} h_{i-1}$ for $i > 0$. Then the bi-infinite tower $(\ldots, f_{-i}^{c'}, \ldots, f_0^{c'} = h_0, \ldots, h_i, \ldots)$ has the combinatorial type $\bar{\sigma}$ and by Theorem 5.1 it must coincide with $\mathcal{T}_{\bar{\sigma}}$. Hence $f_0^{\bar{\sigma}'} = f_0^{\bar{\sigma}}$. To show the continuity of ι, let us assume the contrary. In this case the considerations of compactness following from complex *a priori* bounds imply that there exist two distinct bi-infinite sequences of critical circle maps $(f_i)_{-\infty}^{\infty}$, $(g_i)_{-\infty}^{\infty}$ such that $\sigma(f_i) = \sigma(g_i)$ for every $i \in \mathbb{Z}$, and $f_{i+1} = \mathcal{R}_{\text{cyl}} f_i$ or $f_{i+1} = \mathcal{P}_{\sigma(f_{i+1})} f_i$, and similarly for g_i. Consider first the case, when there exists an infinite sequence $i_k \to -\infty$ such that $\rho(f_{i_k}) \in \mathbb{Q}$. Then f_{i_k} and g_{i_k} are necessarily parabolic, and the existence of such two sequences is ruled out by the Tower Rigidity Theorem. In the complementary case, the existence of two such sequences contradicts the (non-parabolic) Tower Rigidity Theorem of [Ya2].

Convergence to $\mathcal{A}$.
Firstly, observe that the claim (a) is automatically satisfied.

To deal with the claim (b), assume the contrary. Set $f_i = \mathcal{P}_{\sigma_i} \circ \cdots \circ \mathcal{P}_{\sigma_1}(f)$ and similarly for g_i. Then there exists $\epsilon > 0$ and a sequence $\{n_k\}$ such that $\text{dist}(f_{n_k}, g_{n_k}) > \epsilon$. Passing to a further subsequence, we may ensure that $f_{n_k+j} \to \hat{f}_j$, $g_{n_k+j} \to \hat{g}_j$ for all $j \in \mathbb{Z}$. Then $(\hat{f}_j)$, $(\hat{g}_j)$ are distinct bi-infinite sequences of critical circle maps with the same properties as above, their existence is ruled out in the same way.

To prove the claim (c), note that by real *a priori* bounds every sequence of natural numbers has a subsequence j_k such that $\mathcal{R}_{\text{cyl}}^{n_{j_k}+i} f \to g_i$ for $i \leq 0$. By complex *a priori* bounds, whenever g_i is parabolic, with combinatorial type σ, the corresponding immediate basin is proper. By Proposition 2.7, $g_i \in \mathbf{Par}_\sigma$. Therefore, $g_0 \in \mathcal{A}$, and by Theorem 5.1, it is independent of the subsequence j_k, and is equal to $\iota([r_i]_{-\infty}^0)$.

Part (d).

And finally, let us proceed to proving (d). Let us select an annulus U_f around $\mathbb{T}$ so that f is defined in U_f, and U_f compactly contains the immediate basin B of the parabolic orbit of f, and moreover, a there is a subannulus $\mathbb{T} \Subset U \Subset U_f$ whose image under f is the whole of $CC/\mathbb{Z}$. Denote by n the period of the parabolic orbit of f. By the Implicit Function Theorem, there exists an open neighborhood $X \subset \mathbf{A}(U_f)$ of the map f such that

$$G = \{g \in X, \text{ having a parabolic periodic orbit with period } n, \text{ and such that } g'(0) = 0\}$$

is a codimension two analytic submanifold in $\mathbf{A}(U_f)$. Of course, if we denote $G_{\mathbb{R}}$ the real Banach manifold

$$G_{\mathbb{R}} = G \cap \{\bar{g}(z) = g(\bar{z})\},$$

then P is defined on all of $G_{\mathbb{R}}$.

Proposition 8.1. *There exists an open neighborhood Z of f in the submanifold G, such that the operator P continuously extends to an analytic operator $Z \to Z$. Moreover, for each $g \in Z$, the map $h = Pg$ has a period n parabolic point, whose basin $B_h \ni 0$, is compactly contained in U_f, three-fold, and proper. Finally, there is a holomorphic motion $h \to B_h$ over Z.*

Proof. It is not hard to see that P has an analytic extension to a larger neighborhood. Indeed, let $\theta_1\mathbb{T}$ be such that the composition $\tau_{\theta_1} \circ E_f$ has a period n periodic point at 0 with combinatorial type σ_1. By continuity, for a map $g \in G$ close to f, there is $\theta_1(g) \in \mathbb{C}/\mathbb{Z}$ such that $\tau_{\theta_1(g)} \circ E_g$ has the same property. Provided that g is close enough to f, the image $g(U_f)$ compactly contains U_f. This, and the analyticity of the dependence of the eigenvalue of a fixed point of $\tau_\theta \circ E_g$ on θ, implies that there exists $\theta_2(g)$ for which the composition $\tau_{\theta_2(g)} \circ E_g$ has a parabolic orbit of period n. By the Implicit Function Theorem, such $\theta_2(g)$ may be selected analytically in g, and we may set

$$P = \tau_{\theta_2(g)} \circ E_g$$

in the appropriate neighborhood of f. Its analyticity follows from Proposition 2.1.

To establish the remaining assertions it is, however, more convenient to define P in a more explicit fashion. Namely, we first observe, as in the proof of Proposition 6.1, that the boundary of B is a topological repeller, and hence there exists a sub-annulus $A \Supset B$ in U_f such that there is a three-fold preimage $f^{-n}(A) \supset B$ properly contained in A. Therefore, for all maps g in G sufficiently close to f the immediate basin of the parabolic orbit is compactly contained in U_f and is proper. Moreover, its boundary moves holomorphically with g (the holomorphic motion may be constructed, for example, using the preimages of the parabolic orbit, which are dense in ∂B. By Theorem 6.1 each such g has a pinched cubic-like restriction $K(g)$-quasiconformally conjugate to that of f. Let us choose an open (in the induced topology) subset $W \subset G$ such that for $g \in W$ the dilatation $K(g)$ may be bounded by K_0 with the property that for a normalized K_0-quasiconformal map $q : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ the image $q(\mathrm{Dom}(f)) \Supset U_1$ where $U_1 \Supset U_f$. To define $P : W \to W$ consider a hybrid conjugacy between pinched cubic-like restrictions of $g \in W$ and an Arnold's map

f_θ. We may use the conjugacy to perform a quasiconformal surgery transplanting parabolic renormalization $P_{\sigma_1} f_\theta$ to B_g. The desired properties evidently follow. $\Box$

Proposition 8.2. *The differential $D_f(P|_W)$ is compact.*

Proof. The image of a bounded vector field v under $D_f(P|_W)$ is a vector field in $U_1 \Supset W$. By considerations of compactness, it is bounded in W, and the claim follows by the Koebe Theorem. $\Box$

The final point of the argement is:

Proposition 8.3. *The spectral radius*

$$R_{sp}(D_f(P|_W)) < 1.$$

Proof. Observe first that $P^n g \to f$ uniformly for $g \in W$. Indeed, otherwise, there is an $\epsilon > 0$, and sequences $g_k \in W$, $n_k \to \infty$ such that $\operatorname{dist}(P^{n_k} g_k, f) > \epsilon$. Reasoning as in part (b) we find that this contradicts Theorem 5.1. Hence there is an open neighborhood V of f and $N \in \mathbb{N}$ such that $P^N(V) \subset V$. The claim follows by the application of the Banach manifold version of the Schwarz Lemma. $\Box$

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