DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 8, 2016, 1–4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!

Problem 1. Let G be a finite group.

(a) Show that if $H \subsetneqq G$ is a proper subgroup, then there exists an $x \in G$ that is not contained in any subgroup conjugate to H.

(b) A *maximal subgroup* is a proper subgroup $M \subsetneqq G$ that is maximal for this property, i.e., if $M' \subsetneqq G$ is a proper subgroup of G that contains M, then $M' = M$. Show that if all maximal subgroups of G are conjugate, then G is cyclic.

Problem 2. A *chain of prime ideals of length* n in a commutative ring R is an increasing sequence $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$, where each P_i is a prime ideal in R.

(a) Show that if R is a PID, every chain of prime ideals has length 0 or 1.

(b) Exhibit a chain of prime ideals of length 2 in $\mathbb{Z}[x]$.

(c) Find a ring R with a chain of prime ideals of length 2016.

Problem 3.

(a) State the structure theorem for modules over a PID.

(b) Suppose that K is a field and V a K-vector space of dimension 3. How many similarity classes of linear transformations $T : V \to V$ are there that satisfy $T^2(T-1) = 0$? Among them, how many have $\dim \ker(T) = 1$? (Explain how you use part (a)! Also, recall that linear transformations S, T are called *similar* if there is a linear isomorphism $U : V \to V$ such that $S = UTU^{-1}$.)

Problem 4.

(a) Prove or disprove that $f(x) = x^4 + 6x - 3$ is irreducible over the field $\mathbb{Q}(\sqrt[3]{5})$.

(b) Let L be a finite Galois extension of a field K with Galois group $\mathrm{Gal}(L/K)$. Suppose that F is a proper subfield of L that contains K. Prove that $\bigcap_{\sigma \in \mathrm{Gal}(L/K)} \sigma(F)$ is a Galois extension of K. Give complete statements of all results from Galois theory which are used in your solution.

Problem 5. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of a finite group G. For $1 \leq j \leq r$, let ρ_j be an irreducible representation of G whose character equals χ_j.

(a) Prove that if $x \in G$ and $x \neq e$, then there exists j such that $\chi_j(x) \neq \chi_j(e)$.

(b) Let $y \in G$. Prove that if $\rho_j(y)$ is a scalar multiple of the identity operator for all $1 \leq j \leq r$, then y belongs to the centre of G.

Problem 6. Let R be a ring with 1 and let M be a left R-module such that $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where each S_i is a nonzero simple left R-submodule of M.

(a) Prove that any nonzero simple left R-submodule of M is isomorphic to S_i for some i.

(b) What additional conditions on the submodules S_i guarantee that any nonzero simple left R-submodule of M is *equal* to S_i for some i? (Justify your answer.)