Department of Mathematics
University of Toronto

Topology Comprehensive Exam

September 30, 2020, 3:00-6:00 pm

Topology I

Problem 1. Let X be a smooth manifold and

$$\Delta_X = \Big\{(x, x) \mid x \in X\Big\} \subset X \times X$$

be the diagonal. Prove that there exists a natural orientation on some neighborhood N of Δ_X in $X \times X$, whether or not X can be oriented.

Problem 2. Let X be a smooth manifold (with no boundary) of any dimension, and let $f\colon X \to \mathbb{R}^2$ be a smooth map. Prove that, for almost every point $a \in \mathbb{R}$, the horizontal line in $\mathbb{R}^2$ through $(0, a)$ is transversal to f.

Problem 3. Consider two closed loops γ_1 and γ_2 and two closed 1–forms ω_1 and ω_2 on a smooth manifold X. Assume γ_1 is homotopic to γ_2 and ω_1 is cohomologous to ω_2 (their difference is exact). Prove that

$$\int_{\gamma_1} \omega_1 = \int_{\gamma_2} \omega_2.$$

Topology II

Problem 4. Let G be a topological group, i.e. a group which is a topological space such that the group operations are continuous. Let $e \in G$ be the identity element.

Prove that $\pi_1(G, e)$ is abelian.

Problem 5. Let M^n, N^m be topological manifolds of dimensions n and m respectively with $n \neq m$. Prove that M and N are not homeomorphic.

Problem 6.

Does there exist a continuous map of nonzero degree from $\mathbb{S}^2 \times \mathbb{S}^4$ to $\mathbb{CP}^3$? If yes, construct one, if no, prove that no such map exists.