PDE Comprehensive Exam (Fall 2019)

Problem 1. Consider the operator $Lu = 2\frac{\partial^2 u}{\partial x \partial y}$ on the domain

$$U := \{(x, y) \in \mathbb{R}^2 \mid y > x\}.$$

(1) Is this operator elliptic, parabolic or hyperbolic? (HINT: Diagonalize the matrix $a_{ij} = a_{ji}$.)

(2) If we try to prescribe the boundary values of both u and its outward normal derivative $\frac{\partial u}{\partial \nu}$ on the boundary of U to be given, respectively, by functions $g \in C^2(\partial U)$ and $h \in C^1(\partial U)$:

How many solutions does the equation $Lu = 0$ admit?

How smooth will these solutions be in the interior of U?

Problem 2. Assume that U is an open, bounded set of $\mathbb{R}^n$, with smooth boundary.

(1) Let $\lambda_1 > 0$ the smallest eigenvalue of the operator $(-\Delta)$ with zero Dirichlet boundary condition on U. Let u be a smooth solution of the diffusion equation

$$u_t - \Delta u = 0 \text{ in } U \times (0, \infty)$$
$$u = 0 \text{ on } \partial U \times [0, \infty).$$
$$u = g \text{ on } U \times \{t = 0\}.$$

Prove the exponential decay estimate:

$$\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\lambda_1 t} \|g\|_{L^2(U)}.$$

(2) Suppose that u is a smooth solution of

$$u_t - \Delta u + q(x)u = 0 \text{ in } U \times (0, \infty)$$
$$u = 0 \text{ on } \partial U \times [0, \infty).$$
$$u = g \text{ on } U \times \{t = 0\}.$$

and the function q satisfies for all $x \in U$, $q(x) \geq \alpha > 0$, (α constant). Prove

$$|u(x, t)| \leq Ce^{-\alpha t} \quad (x, t) \in U \times (0, T).$$

Problem 3. Consider the solution $u = u(x, t)$ of the quasilinear partial differential equation

$$u_t + a(u)u_x = 0, \qquad u(x, 0) = f(x) \tag{1}$$

(1) Derive an implicit formula for the solution u.

(2) Show that u becomes singular for some $t > 0$ unless $a(f(s))$ is a non-decreasing function of s.

(3) Define the concept of a *classical* and a *weak* (or integral) solution of (1).

Problem 4. Fix a bounded domain $U \Subset \mathbb{R}^n$ with C^1 smooth boundary, and $f \in L^\infty(U)$. Define the functional

$$E(u) := \int_U \left(\frac{1}{2}|Du|^2 - fu \right) dx.$$

(1) If the functional $E(u)$ happens to be minimized on $W_0^{1,2}(U)$ by some $u \in C^2(\overline{U})$, derive the partial differential equation that will be satisfied by u.

(2) Show the partial differential equation derived in part (1) has at most one solution in $C^2(\overline{U})$ satisfying the boundary condition $u = 0$ on ∂U.

(3) Show the functional $E(u)$ has a unique minimizer in $W_0^{1,2}(U)$.

Problem 5. Let $u = u(x,t)$ be a solution of the 3+1 dimensional nonlinear Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + u = u^3 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u = g, \quad u_t = h & \text{on} \quad \mathbb{R}^3 \times \{0\}. \end{cases} \tag{KG}$$

where the data (g,h) are smooth and compactly supported.

(1) Define the energy functional

$$\mathcal{E}(u)(t) = \frac{1}{2}\|\partial_t u(t)\|^2_{H^2(\mathbb{R}^3)} + \frac{1}{2}\|\nabla u(t)\|^2_{H^2(\mathbb{R}^3)} + \frac{1}{2}\|u(t)\|^2_{H^2(\mathbb{R}^3)}.$$

Prove that solutions of (KG) satisfy, for some constant C, the energy inequality

$$\frac{d}{dt}\mathcal{E}(u)(t) \le C\big(\mathcal{E}(u)(t)\big)^2.$$

(2) Deduce that on some time interval $[0,T]$ for T small enough depending on the energy of the initial data $\mathcal{E}(u)(0)$, one has the estimate

$$\mathcal{E}(u)(t) \le 2\mathcal{E}(u)(0).$$

Problem 6. Consider the initial value problem associated to the linearized Korteweg-de Vries equation

$$\begin{aligned} &\partial_t u + \partial_{xxx} u = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ &u(x,0) = u_0(x). \end{aligned}$$

Here $u(x,t)$ is real-valued. Assume $u \in \mathbb{S}(\mathbb{R})$ (i.e. of Schwartz class).

(1) Write the differential equation satisfied by $\hat{u}(k,t)$, the Fourier transform of u (in x), and solve it.

(2) Write u in the form of a convolution of u_0 with a kernel $T(x,t)$, written in terms of the Airy function

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(kx + \frac{k^3}{3})} dk.$$