DEPARTMENT OF MATHEMATICS
University of Toronto

Comprehensive Exam, 2021

First half: Monday, September 27, 3-6pm, in MP137

Time: 3 hours. Please be brief but justify your work
If you appeal to a standard result, be sure to carefully quote it (and verify the assumptions)

Passing score: 6/12 (over 2 days)
Do not attempt all problems; instead, aim for complete solutions

1. (a) Suppose that G is a (non-trivial) finite group and let p be the smallest prime divisor of the order of G. Show that any normal subgroup of G of order p is contained in the centre of G.

   (b) Suppose that G is a finite *simple* group and p a prime number such that p^2 divides the order of G. Show that any proper subgroup H of G has index $(G : H)$ at least $2p$. (Hint: use a suitable group action.)

   (c) In the setting of part (b), give an example where equality can hold when $G = A_6$ (i.e. find p and H such that p^2 divides $|G|$ and $(G : H) = 2p$).

2. Prove that every discrete normal subgroup of a connected topological group is abelian.

3. Prove the following version of the Dominated Convergence Theorem: Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f, f_1, f_2, \ldots$ be measurable functions on $(X, \mathcal{M})$. If

   - $f_n \to f$ for μ a.e. x,

   and there exist $g, g_1, g_2, \cdots \in L^1(X, \mu)$ such that

   - $|f_n| \le |g_n|$, and
   - $\|g_n - g\|_1 \to 0$ as $n \to \infty$,

   then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

4. The purpose of this problem is to prove that, if $P(z)$ is a non-constant complex polynomial, then the zeros of $P'(z)$ lie in the convex hull of the set of zeros of P.

   (a) Suppose that $P(z)$ has degree $n \geq 1$ and zeros $b_1, \ldots, b_n$ (each zero listed as many times as its multiplicity). Show that

   $$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - b_k}.$$

   (b) Show that, if $P'(z) = 0$, then

   $$\left(\sum_{k=1}^{n} \frac{1}{|z - b_k|^2} \right) \overline{z} = \sum_{k=1}^{n} \frac{\overline{b}_k}{|z - b_k|^2}.$$

   (c) Deduce that, if $P'(z) = 0$, then z lies in the convex hull of the points b_k.

5. Let S^2 be the 2-sphere and $A \subset S^2$ a subset of cardinality 3. We denote by S^2/A the space obtained by contracting A to a point.

   (a) What is the fundamental group of S^2/A?

   (b) Compute the singular cohomology groups $H^*(S^2/A; \mathbf{Z})$ and the cup product structure.

   Justify your answers with proofs.

6. Let $G = \mathrm{Gal}(K/F)$ be the Galois group of the splitting field of a monic, integral polynomial $f(x) \in \mathbb{Z}[x]$ of degree n.

   (a) Prove that G acts by permutation on the roots of $f(x)$ in K.

   (b) Under what condition on G is $f(x)$ irreducible over $\mathbb{Q}$?

   (c) More generally, under what conditions on G do the irreducible factors of f have degrees $(n_1, \ldots, n_r)$?

   (d) What about irreducibility over $\mathbb{Z}$?