DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Comprehensive Examination

Wednesday, September 4, 2019   120 minutes

1. (a) Prove that a nonconstant holomorphic mapping is *open* (i.e., the image of every open set is open).

   (b) Let U, V denote domains in $\mathbb{C}$ and let $f : U \to V$ be a holomorphic mapping. Suppose that f is *proper* (i.e., $f^{-1}(K)$ is compact, for every compact subset K of V). Prove that $f(U) = V$.

   (c) Is the assertion in (a) true if "holomorphic" is replaced by "continuous"? Explain.

2. Let $\mathcal{A}$ denote the set of all holomorphic functions $f(z)$ on the open unit disk $D = \{|z| < 1\}$ such that $f(0) = 1$ and $\operatorname{Re} f > 0$.

   (a) Show that, if $f \in \mathcal{A}$, then
   $$\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}\,.$$
   (Hint. Schwarz's Lemma.)

   (b) Prove that $\mathcal{A}$ is a normal family.

   (c) How large can $|f'(0)|$ be?

3. Use residues to show that
   $$\int_0^1 \frac{dx}{\sqrt[3]{x^2 - x^3}} = \frac{2\pi}{\sqrt{3}}\,.$$

4. (a) Give an example of a nonconstant meromorphic function on $\mathbb{C}$ that omits *two* values.

   (b) Use Picard's little theorem (for entire functions) to prove *Picard's little theorem for meromorphic functions*: Every meromorphic function on $\mathbb{C}$ that omits three distinct values is constant.